Supplemental Material
Generalized Hydrodynamics Revisited

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I. MICROSCOPIC CONSERVATION LAWS

The quantum mechanical microscopic conservation laws for the number, momentum, and energy density operators are given in Refs. 1 and 2. However, the details of the derivation are not given and the results do not include an external force. For completeness the general derivation is given here.

The time dependence of an operator $A(t)$ which depends on the position and momenta of the system particles is given by

$$\frac{\partial}{\partial t}A(t) = i\left[H_N(t), A(t)\right], \tag{1}$$

where the Hamiltonian of interest is of the form

$$H_N(t) = \sum_{\alpha=1}^{N}\left(\frac{p_\alpha^2}{2m} + v^{\text{ext}}(\mathbf{q}_\alpha, t)\right) + \frac{1}{2}\sum_{\alpha\neq\beta=1}^{N} U\left(|\mathbf{q}_\alpha - \mathbf{q}_\beta|\right). \tag{2}$$

Associated with the continuous symmetries are the usual conservation laws for particle number, linear momentum, energy, and angular momentum. Here only point particles are considered so the relevant conservation laws are those of particle number, momentum, and energy. The associated local densities are defined by

$$n(\mathbf{r}) = \sum_{\alpha=1}^{N}\delta(\mathbf{r} - \mathbf{q}_\alpha), \tag{3}$$

$$\mathbf{p}(\mathbf{r}) = \sum_{\alpha=1}^{N}\frac{1}{2}\left[\mathbf{p}_\alpha, \delta(\mathbf{r} - \mathbf{q}_\alpha)\right]_+, \tag{4}$$

$$e(\mathbf{r}) = \sum_{\alpha=1}^{N}\frac{1}{2}\left[\frac{p_\alpha^2}{2m}, \delta(\mathbf{r} - \mathbf{q}_\alpha)\right]_+ + \frac{1}{2}\sum_{\alpha\neq\beta=1}^{N} U(|\mathbf{q}_\alpha - \mathbf{q}_\beta|)\delta(\mathbf{r} - \mathbf{q}_\alpha), \tag{5}$$

where $[A, B]_+ = AB + BA$ is the anticommutator. The symmetrized products ensure the Hermitian nature of these local density operators. In Eq. (2) $v^{\text{ext}}(\mathbf{q}_\alpha, t)$ is an external potential. As this external potential has not been included in the definition of $e(\mathbf{r})$ in Eq. (5), $e(\mathbf{r})$ is referred to as the "intrinsic" energy density.

The following identities will be used below:

$$[\mathbf{p}_\alpha, f(\mathbf{q}_\alpha)] = -i\frac{\partial f(\mathbf{q}_\alpha)}{\partial \mathbf{q}_\alpha}, \tag{6}$$

$$\left[p_\alpha^2, A(\mathbf{q}_\alpha)\right] = -i\left[p_{\alpha j}, \frac{\partial}{\partial q_{\alpha j}}A(\mathbf{q}_\alpha)\right]_+, \tag{7}$$

$$\left[[A, B]_+, C\right] = [[B, C], A]_+ + [[A, C], B]_+, \tag{8}$$

$$[\mathbf{p}_\alpha, F(\mathbf{q}_\alpha)G(\mathbf{q}_\alpha)]_+ = \frac{1}{2}\left[[\mathbf{p}_\alpha, F(\mathbf{q}_\alpha)]_+, G(\mathbf{q}_\alpha)\right]_+. \tag{9}$$

A. Number conservation

Local conservation laws follow exactly from the Hamiltonian dynamics. The simplest is the conservation of number density. In all derivations below the Greek letters, $\alpha, \beta, \gamma, \cdots$ index the particles while Latin indices $i, j, k, \cdots$ denote Cartesian coordinates. Unless noted, Einstein summation is assumed for repeated indices.

Beginning with the time derivative of the number density,

$$\begin{aligned} i\frac{\partial}{\partial t}n(\mathbf{r}) = [n(\mathbf{r}), H_N(t)] &= \left[\sum_{\alpha=1}^{N}\delta(\mathbf{r}-\mathbf{q}_\alpha), \sum_{\beta=1}^{N}\frac{p_\beta^2}{2m}\right] \\ &= \frac{1}{2m}\sum_{\alpha=1}^{N}\left[\delta(\mathbf{r}-\mathbf{q}_\alpha), p_\alpha^2\right]. \end{aligned} \tag{10}$$

Using Eq. (7),

$$\begin{aligned} i\frac{\partial}{\partial t}n(\mathbf{r}) &= \frac{i}{2m}\sum_{\alpha=1}^{N}\left(p_{\alpha j}\frac{\partial}{\partial q_{\alpha j}}\delta(\mathbf{r}-\mathbf{q}_\alpha) + \frac{\partial}{\partial q_{\alpha j}}\delta(\mathbf{r}-\mathbf{q}_\alpha)p_{\alpha j}\right) \\ &= -\frac{i}{2m}\sum_{\alpha=1}^{N}\left(p_{\alpha j}\frac{\partial}{\partial r_j}\delta(\mathbf{r}-\mathbf{q}_\alpha) + \frac{\partial}{\partial r_j}\delta(\mathbf{r}-\mathbf{q}_\alpha)p_{\alpha j}\right) \\ &= -\frac{i}{m}\frac{\partial}{\partial r_j}\sum_{\alpha=1}^{N}\frac{1}{2}\left[p_{\alpha j}, \delta(\mathbf{r}-\mathbf{q}_\alpha)\right]_+. \end{aligned} \tag{11}$$

Inserting the definition of the momentum density in Eq. (4) gives the final result,

$$\frac{\partial}{\partial t}n(\mathbf{r}) + \frac{1}{m}\nabla\cdot\mathbf{p}(\mathbf{r}) = 0. \tag{12}$$

B. Momentum conservation

The time derivative of $\mathbf{p}(\mathbf{r}, t)$ is

$$i\frac{\partial}{\partial t}\mathbf{p}(\mathbf{r}) = [\mathbf{p}(\mathbf{r}), H_N(t)], \tag{13}$$

$$i\frac{\partial}{\partial t}p_j(\mathbf{r}) = \frac{1}{2}\sum_{\alpha=1}^{N}\left[\left[p_{\alpha j}, \delta(\mathbf{r}-\mathbf{q}_\alpha)\right]_+, H_N(t)\right]. \tag{14}$$

Using $\left[[A,B]_+, C\right] = [[B,C], A]_+ + [[A,C], B]_+$ from Eq. (8), this is

$$i\frac{\partial}{\partial t}p_j(\mathbf{r}) = \frac{1}{2}\sum_{\alpha=1}^{N}\Big([[\delta(\mathbf{r}-\mathbf{q}_\alpha), H_N(t)], p_{\alpha j}]_+ + [[p_{\alpha j}, H_N(t)], \delta(\mathbf{r}-\mathbf{q}_\alpha)]_+\Big). \tag{15}$$

The two commutators appearing in the above equation will be treated separately. The former is

$$\begin{aligned} [\delta(\mathbf{r}-\mathbf{q}_\alpha), H_N] &= \frac{1}{2m}\left[\delta(\mathbf{r}-\mathbf{q}_\alpha), p_\alpha^2\right] \\ &= \frac{i}{2m}\left[p_{\alpha k}, \frac{\partial}{\partial q_{\alpha k}}\delta(\mathbf{r}-\mathbf{q}_\alpha)\right]_+ \\ &= \frac{1}{2m}\left(-i\frac{\partial}{\partial r_k}\right)[\delta(\mathbf{r}-\mathbf{q}_\alpha), p_{\alpha k}]_+. \end{aligned} \tag{16}$$

The other commutator is

$$[p_{\alpha j}, H_N] = \left[p_{\alpha j}, \left(v^{\rm ext}(\mathbf{q}_\alpha, t) + \frac{1}{2}\sum_{\beta\neq\alpha}\big(U(|\mathbf{q}_\alpha - \mathbf{q}_\beta|) + U(|\mathbf{q}_\beta - \mathbf{q}_\alpha|)\big)\right)\right]$$
$$= -i\frac{\partial}{\partial q_{\alpha j}} v^{\rm ext}(\mathbf{q}_\alpha, t) - i\frac{\partial}{\partial q_{\alpha j}}\sum_{\beta\neq\alpha} U(|\mathbf{q}_\alpha - \mathbf{q}_\beta|)$$
$$= iF^{\rm ext}_{\alpha j}(\mathbf{q}_\alpha, t) + i\sum_{\beta\neq\alpha} F_{\alpha\beta j}(|\mathbf{q}_\alpha - \mathbf{q}_\beta|), \tag{17}$$

where

$$F^{\rm ext}_{\alpha j}(\mathbf{q}_\alpha, t) = -\frac{\partial}{\partial q_{\alpha j}} v^{\rm ext}(\mathbf{q}_\alpha, t) \tag{18}$$

and

$$F_{\alpha\beta j}(|\mathbf{q}_\alpha - \mathbf{q}_\beta|) = -\frac{\partial}{\partial q_{\alpha j}} U(|\mathbf{q}_\alpha - \mathbf{q}_\beta|). \tag{19}$$

Using Eqs. (16) and (17) in Eq. (15) yields

$$\frac{\partial}{\partial t} p_j(\mathbf{r}) = \frac{1}{2}\sum_{\alpha=1}^{N}\left(\left[-\frac{1}{2m}\frac{\partial}{\partial r_k}\left[\delta(\mathbf{r} - \mathbf{q}_\alpha), p_{\alpha k}\right]_+, p_{\alpha j}\right]_+ + \left[\left(\sum_{\beta\neq\alpha} F_{\alpha\beta j}(|\mathbf{q}_\alpha - \mathbf{q}_\beta|) + F^{\rm ext}_{\alpha j}(\mathbf{q}_\alpha, t)\right), \delta(\mathbf{r} - \mathbf{q}_\alpha)\right]_+\right). \tag{20}$$

Define the kinetic part of the total momentum flux to be

$$t^K_{jk}(\mathbf{r}, t) = \frac{1}{4m}\sum_{\alpha=1}^{N}\left[p_{\alpha j}, \left[p_{\alpha k}, \delta(\mathbf{r} - \mathbf{q}_\alpha)\right]_+\right]_+ \tag{21}$$

so

$$\frac{\partial}{\partial t} p_j(\mathbf{r}) = -\frac{\partial}{\partial r_k} t^K_{jk}(\mathbf{r}, t) + \frac{1}{2}\sum_{\alpha=1}^{N}\left[\left(\sum_{\beta\neq\alpha} F_{\alpha\beta j}(|\mathbf{q}_\alpha - \mathbf{q}_\beta|) + F^{\rm ext}_{\alpha j}(\mathbf{q}_\alpha)\right), \delta(\mathbf{r} - \mathbf{q}_\alpha)\right]_+$$
$$= -\frac{\partial}{\partial r_k} t^K_{jk}(\mathbf{r}, t) + \sum_{\alpha\neq\beta=1}^{N} F_{\alpha\beta j}(|\mathbf{q}_\alpha - \mathbf{q}_\beta|)\delta(\mathbf{r} - \mathbf{q}_\alpha) + \sum_{\alpha=1}^{N} F^{\rm ext}_{\alpha j}(\mathbf{q}_\alpha)\delta(\mathbf{r} - \mathbf{q}_\alpha). \tag{22}$$

The force density arising from the external potential is defined by

$$f^{\rm ext}_j(\mathbf{r}, t) = \sum_{\alpha=1}^{N} F^{\rm ext}_{\alpha j}(\mathbf{q}_\alpha)\delta(\mathbf{r} - \mathbf{q}_\alpha) \tag{23}$$

giving

$$\frac{\partial}{\partial t} p_j(\mathbf{r}) = -\frac{\partial}{\partial r_k} t^K_{jk}(\mathbf{r}, t) + \sum_{\alpha\neq\beta=1}^{N} F_{\alpha\beta j}(|\mathbf{q}_\alpha - \mathbf{q}_\beta|)\delta(\mathbf{r} - \mathbf{q}_\alpha) + f^{\rm ext}_j(\mathbf{r}, t). \tag{24}$$

The second term on the right side requires some care. First, interchanging the dummy labels α, β and using Newton's third law gives

$$\sum_{\alpha\neq\beta=1}^{N} F_{\alpha\beta j}(|\mathbf{q}_\alpha - \mathbf{q}_\beta|)\delta(\mathbf{r} - \mathbf{q}_\alpha) = \sum_{\alpha\neq\beta=1}^{N} F_{\beta\alpha j}(|\mathbf{q}_\alpha - \mathbf{q}_\beta|)\delta(\mathbf{r} - \mathbf{q}_\beta)$$
$$= -\sum_{\alpha\neq\beta=1}^{N} F_{\alpha\beta j}(|\mathbf{q}_\alpha - \mathbf{q}_\beta|)\delta(\mathbf{r} - \mathbf{q}_\beta). \tag{25}$$

Taking half of the sum of these two equivalent expressions gives

$$\sum_{\alpha\neq\beta=1}^{N} F_{\alpha\beta j}(|\mathbf{q}_\alpha-\mathbf{q}_\beta|)\delta(\mathbf{r}-\mathbf{q}_\alpha)=\frac{1}{2}\sum_{\alpha\neq\beta=1}^{N} F_{\alpha\beta j}(|\mathbf{q}_\alpha-\mathbf{q}_\beta|)\left(\delta(\mathbf{r}-\mathbf{q}_\alpha)-\delta(\mathbf{r}-\mathbf{q}_\beta)\right). \tag{26}$$

Consider a parameter λ defining an arbitrary path from $\mathbf{x}(\lambda_\beta)=\mathbf{q}_\beta$ to $\mathbf{x}(\lambda_\alpha)=\mathbf{q}_\alpha$. The following identity then holds,

$$\delta(\mathbf{r}-\mathbf{q}_\alpha)-\delta(\mathbf{r}-\mathbf{q}_\beta)=\int_{\lambda_\beta}^{\lambda_\alpha} d\lambda\ \frac{d}{d\lambda}\delta(\mathbf{r}-\mathbf{x}(\lambda)) \tag{27}$$

so that

$$\begin{aligned}\delta(\mathbf{r}-\mathbf{q}_\alpha)-\delta(\mathbf{r}-\mathbf{q}_\beta)&=\int_{\lambda_\beta}^{\lambda_\alpha} d\lambda\ \frac{dx_k}{d\lambda}\frac{d}{dx_k}\delta(\mathbf{r}-\mathbf{x}(\lambda))\\&=-\frac{d}{dr_k}\int_{\lambda_\beta}^{\lambda_\alpha} d\lambda\ \frac{dx_k}{d\lambda}\delta(\mathbf{r}-\mathbf{x}(\lambda))\\&\equiv-\frac{\partial}{\partial r_k}\mathcal{D}_k(\mathbf{r},\mathbf{q}_\alpha,\mathbf{q}_\beta).\end{aligned} \tag{28}$$

Thus the interaction force term becomes

$$\begin{aligned}\sum_{\alpha\neq\beta=1}^{N} F_{\alpha\beta j}(|\mathbf{q}_\alpha-\mathbf{q}_\beta|)\delta(\mathbf{r}-\mathbf{q}_\alpha)&=-\frac{\partial}{\partial r_k}\frac{1}{2}\sum_{\alpha\neq\beta=1}^{N} F_{\alpha\beta j}(|\mathbf{q}_\alpha-\mathbf{q}_\beta|)\mathcal{D}_k(\mathbf{r},\mathbf{q}_\alpha,\mathbf{q}_\beta)\\&=-\frac{\partial}{\partial r_k}t^P_{jk}(\mathbf{r},t)\end{aligned} \tag{29}$$

where the potential part of the momentum flux density is

$$t^P_{jk}(\mathbf{r},t)\equiv\frac{1}{2}\sum_{\alpha\neq\beta=1}^{N} F_{\alpha\beta j}(|\mathbf{q}_\alpha-\mathbf{q}_\beta|)\mathcal{D}_k(\mathbf{r},\mathbf{q}_\alpha,\mathbf{q}_\beta). \tag{30}$$

Using Eq. (29) in Eq. (24) gives

$$\frac{\partial}{\partial t}p_j(\mathbf{r})=-\frac{\partial}{\partial r_k}\left(t^K_{jk}(\mathbf{r},t)+t^P_{jk}(\mathbf{r},t)\right)+f^{\mathrm{ext}}_j(\mathbf{r},t). \tag{31}$$

Defining the total momentum flux density as the sum of its kinetic and potential parts, $t_{jk}=t^K_{jk}+t^P_{jk}$, gives a compact form for the conservation of momentum equation,

$$\frac{\partial}{\partial t}p_j(\mathbf{r})+\frac{\partial}{\partial r_k}t_{jk}(\mathbf{r},t)=f^{\mathrm{ext}}_j(\mathbf{r},t). \tag{32}$$

C. Energy Conservation

The conservation of energy equation also follows from the time derivative of the local energy density, using Eqs. (2) and (5),

$$i\frac{\partial}{\partial t}e(\mathbf{r},t)=[e(\mathbf{r}),H_N(t)] \tag{33}$$

$$i\frac{\partial}{\partial t}e(\mathbf{r},t)=\left[\sum_{\alpha=1}^{N}\frac{1}{2}\left[\frac{p_\alpha^2}{2m},\delta(\mathbf{r}-\mathbf{q}_\alpha)\right]_+,H_N(t)\right]+\left[\frac{1}{2}\sum_{\alpha\neq\beta=1}^{N}U(|\mathbf{q}_\alpha-\mathbf{q}_\beta|)\delta(\mathbf{r}-\mathbf{q}_\alpha),H_N(t)\right]. \tag{34}$$

The following two sections deal separately with each commutator appearing in Eq. (34).

1. First term of Eq. (34)

The first commutator can be written as the sum of three terms,

$$\begin{aligned}
&\left[\sum_{\alpha=1}^{N}\frac{1}{2}\left[\frac{p_\alpha^2}{2m},\delta(\mathbf{r}-\mathbf{q}_\alpha)\right]_+,H_N(t)\right]\\
&=\frac{1}{4m}\sum_{\alpha=1}^{N}\left[\left[\delta(\mathbf{r}-\mathbf{q}_\alpha),H_N(t)\right],p_\alpha^2\right]_+ +\frac{1}{4m}\left[\left[p_\alpha^2,H_N(t)\right],\delta(\mathbf{r}-\mathbf{q}_\alpha)\right]_+\\
&=\frac{1}{8m^2}\sum_{\alpha=1}^{N}\left[\left[\delta(\mathbf{r}-\mathbf{q}_\alpha),p_\alpha^2\right],p_\alpha^2\right]_+\\
&\quad-\frac{i}{4m}\sum_{\alpha=1}^{N}\left[\left[p_{\alpha j},\frac{\partial}{\partial q_{\alpha j}}\left(v^{\text{ext}}(\mathbf{q}_\alpha,t)+\frac{1}{2}\sum_{\beta\neq\alpha}\left(U(|\mathbf{q}_\alpha-\mathbf{q}_\beta|)+U(|\mathbf{q}_\beta-\mathbf{q}_\alpha|)\right)\right)\right]_+,\delta(\mathbf{r}-\mathbf{q}_\alpha)\right]_+\\
&=\frac{1}{8m^2}\sum_{\alpha=1}^{N}\left[p_\alpha^2,\left[\delta(\mathbf{r}-\mathbf{q}_\alpha),p_\alpha^2\right]\right]_+\\
&\quad-\frac{i}{4m}\sum_{\alpha=1}^{N}\left[\left[p_{\alpha j},\frac{\partial}{\partial q_{\alpha j}}\left(v^{\text{ext}}(\mathbf{q}_\alpha,t)+\sum_{\beta\neq\alpha}U(|\mathbf{q}_\alpha-\mathbf{q}_\beta|)\right)\right]_+,\delta(\mathbf{r}-\mathbf{q}_\alpha)\right]_+\\
&=\frac{i}{8m^2}\sum_{\alpha=1}^{N}\left[p_\alpha^2,\left[p_{\alpha j},\frac{\partial}{\partial q_{\alpha j}}\delta(\mathbf{r}-\mathbf{q}_\alpha)\right]_+\right]_+\\
&\quad-\frac{i}{4m}\sum_{\alpha=1}^{N}\left[-\left[p_{\alpha j},\left(\sum_{\beta\neq\alpha}F_{\alpha\beta j}(|\mathbf{q}_\alpha-\mathbf{q}_\beta|)+F_{\alpha j}^{\text{ext}}(\mathbf{q}_\alpha,t)\right)\right]_+,\delta(\mathbf{r}-\mathbf{q}_\alpha)\right]_+\\
&=-\frac{i}{8m^2}\frac{\partial}{\partial r_j}\sum_{\alpha=1}^{N}\left[p_\alpha^2,\left[p_{\alpha j},\delta(\mathbf{r}-\mathbf{q}_\alpha)\right]_+\right]_+\\
&\quad+\frac{i}{4m}\sum_{\alpha\neq\beta=1}^{N}\left[\left[p_{\alpha j},F_{\alpha\beta j}(|\mathbf{q}_\alpha-\mathbf{q}_\beta|)\right]_+,\delta(\mathbf{r}-\mathbf{q}_\alpha)\right]_+ +\frac{i}{4m}\sum_{\alpha=1}^{N}\left[\left[p_{\alpha j},F_{\alpha j}^{\text{ext}}(\mathbf{q}_\alpha,t)\right]_+,\delta(\mathbf{r}-\mathbf{q}_\alpha)\right]_+ .
\end{aligned}\tag{35}$$

2. Second term of Eq. (34)

Returning to the final commutator of Eq.(34), use the relationship $[p_\alpha, A(\mathbf{q}_\alpha)\delta(\mathbf{r}-\mathbf{q}_\alpha)]_+ = \frac{1}{2}\left[[p_\alpha, A(\mathbf{q}_\alpha)]_+, \delta(\mathbf{r}-\mathbf{q}_\alpha)\right]_+$ from Eq. (9) in several places in the following,

$$
\begin{aligned}
&\left[\frac{1}{2}\sum_{\alpha\neq\beta=1}^{N} U(|\mathbf{q}_\alpha-\mathbf{q}_\beta|)\delta(\mathbf{r}-\mathbf{q}_\alpha), H_N\left(t\right)\right] \\
&\qquad = \left[\frac{1}{2}\sum_{\alpha\neq\beta=1}^{N} U(|\mathbf{q}_\alpha-\mathbf{q}_\beta|)\delta(\mathbf{r}-\mathbf{q}_\alpha), \frac{p_\alpha^2+p_\beta^2}{2m}\right] \\
&\qquad = i\frac{1}{4m}\sum_{\alpha\neq\beta=1}^{N}\left(\left[p_{\alpha j}, \frac{\partial}{\partial q_{\alpha j}}\left(U(|\mathbf{q}_\alpha-\mathbf{q}_\beta|)\delta(\mathbf{r}-\mathbf{q}_\alpha)\right)\right]_+ + \left[p_{\beta j}, \frac{\partial}{\partial q_{\beta j}}\left(U(|\mathbf{q}_\alpha-\mathbf{q}_\beta|)\delta(\mathbf{r}-\mathbf{q}_\alpha)\right)\right]_+\right) \\
&\qquad = -i\frac{1}{4m}\sum_{\alpha\neq\beta=1}^{N}\Big(\left[p_{\alpha j}, F_{\alpha\beta j}(\left|\mathbf{q}_\alpha-\mathbf{q}_\beta\right|)\delta(\mathbf{r}-\mathbf{q}_\alpha)\right]_+ + \left[p_{\beta j}, F_{\beta\alpha j}(\left|\mathbf{q}_\alpha-\mathbf{q}_\beta\right|)\delta(\mathbf{r}-\mathbf{q}_\alpha)\right]_+ \\
&\qquad\qquad - \left[p_{\alpha j}, U(|\mathbf{q}_\alpha-\mathbf{q}_\beta|)\frac{\partial}{\partial q_{\alpha j}}\delta(\mathbf{r}-\mathbf{q}_\alpha)\right]_+\Big) \qquad (36) \\
&\qquad = -i\frac{1}{4m}\sum_{\alpha\neq\beta=1}^{N}\left(\left[\left(p_{\alpha j}-p_{\beta j}\right), F_{\alpha\beta j}(\left|\mathbf{q}_\alpha-\mathbf{q}_\beta\right|)\delta(\mathbf{r}-\mathbf{q}_\alpha)\right]_+ - \frac{\partial}{\partial r_j}\left[p_{\alpha j}, U(|\mathbf{q}_\alpha-\mathbf{q}_\beta|)\delta(\mathbf{r}-\mathbf{q}_\alpha)\right]_+\right) \qquad (37) \\
&\qquad = -\frac{i}{8m}\sum_{\alpha\neq\beta=1}^{N}\left[\left[p_{\alpha j}-p_{\beta j}, F_{\alpha\beta j}(\left|\mathbf{q}_\alpha-\mathbf{q}_\beta\right|)\right]_+, \delta(\mathbf{r}-\mathbf{q}_\alpha)\right]_+ \\
&\qquad\qquad + \frac{\partial}{\partial r_j}\frac{i}{4m}\sum_{\alpha\neq\beta=1}^{N}\left[p_{\alpha j}, U(|\mathbf{q}_\alpha-\mathbf{q}_\beta|)\delta(\mathbf{r}-\mathbf{q}_\alpha)\right]_+ . \qquad (38)
\end{aligned}
$$

In going from Eq. (36) to (37), Newton's third law has been used,

$$
F_{\beta\alpha j}\left(\left|\mathbf{q}_\alpha-\mathbf{q}_\beta\right|\right) = -F_{\alpha\beta j}\left(\left|\mathbf{q}_\alpha-\mathbf{q}_\beta\right|\right). \qquad (39)
$$

3. Combining results

Using Eqs. (35) and (38) in Eq. (34),

$$
\begin{aligned}
i\frac{\partial}{\partial t}e(\mathbf{r},t) = &-\frac{i}{8m^2}\frac{\partial}{\partial r_j}\sum_{\alpha=1}^{N}\left[p_\alpha^2, \left[p_{\alpha j}, \delta(\mathbf{r}-\mathbf{q}_\alpha)\right]_+\right]_+ \\
&+\frac{i}{4m}\sum_{\alpha=1}^{N}\left[\left[p_{\alpha j}, F_{\alpha j}^{\text{ext}}(\mathbf{q}_\alpha,t)\right]_+, \delta(\mathbf{r}-\mathbf{q}_\alpha)\right]_+ \\
&+\frac{i}{4m}\sum_{\alpha\neq\beta=1}^{N}\left[\left[p_{\alpha j}, F_{\alpha\beta j}(|\mathbf{q}_\alpha-\mathbf{q}_\beta|)\right]_+, \delta(\mathbf{r}-\mathbf{q}_\alpha)\right]_+ \\
&-\frac{i}{8m}\sum_{\alpha\neq\beta=1}^{N}\left[\left[p_{\alpha j}-p_{\beta j}, F_{\alpha\beta j}(\left|\mathbf{q}_\alpha-\mathbf{q}_\beta\right|)\right]_+, \delta(\mathbf{r}-\mathbf{q}_\alpha)\right]_+ \\
&+\frac{\partial}{\partial r_j}\frac{i}{4m}\sum_{\alpha\neq\beta=1}^{N}\left[p_{\alpha j}, U(|\mathbf{q}_\alpha-\mathbf{q}_\beta|)\delta(\mathbf{r}-\mathbf{q}_\alpha)\right]_+ . \qquad (40)
\end{aligned}
$$

The terms in the third and fourth lines can be combined,

$$
\begin{aligned}
i\frac{\partial}{\partial t}e(\mathbf{r},t) = &-\frac{i}{8m^2}\frac{\partial}{\partial r_j}\sum_{\alpha=1}^{N}\Big[p_\alpha^2,[p_{\alpha j},\delta(\mathbf{r}-\mathbf{q}_\alpha)]_+\Big]_+ \\
&+\frac{i}{4m}\sum_{\alpha=1}^{N}\Big[\big[p_{\alpha j},F_{\alpha j}^{\rm ext}(\mathbf{q}_\alpha,t)\big]_+,\delta(\mathbf{r}-\mathbf{q}_\alpha)\Big]_+ \\
&+\frac{i}{8m}\sum_{\alpha\neq\beta=1}^{N}\Big[\big[(p_{\alpha j}+p_{\beta j}),F_{\alpha\beta j}(|\mathbf{q}_\alpha-\mathbf{q}_\beta|)\big]_+,\delta(\mathbf{r}-\mathbf{q}_\alpha)\Big]_+ \\
&+\frac{\partial}{\partial r_j}\frac{i}{4m}\sum_{\alpha\neq\beta=1}^{N}\big[p_{\alpha j},U(|\mathbf{q}_\alpha-\mathbf{q}_\beta|)\delta(\mathbf{r}-\mathbf{q}_\alpha)\big]_+ .
\end{aligned}
\tag{41}
$$

The third term can further be rewritten as

$$
\begin{aligned}
&\frac{i}{8m}\sum_{\alpha\neq\beta=1}^{N}\Big[\big[(p_{\alpha j}+p_{\beta j}),F_{\alpha\beta j}(|\mathbf{q}_\alpha-\mathbf{q}_\beta|)\big]_+,\delta(\mathbf{r}-\mathbf{q}_\alpha)\Big]_+ \\
&\quad=\frac{i}{8m}\sum_{\alpha\neq\beta=1}^{N}\left(\Big[\big[p_{\alpha j},F_{\alpha\beta j}(|\mathbf{q}_\alpha-\mathbf{q}_\beta|)\big]_+,\delta(\mathbf{r}-\mathbf{q}_\alpha)\Big]_+ + \Big[\big[p_{\alpha j},F_{\beta\alpha j}(|\mathbf{q}_\alpha-\mathbf{q}_\beta|)\big]_+,\delta(\mathbf{r}-\mathbf{q}_\beta)\Big]_+\right) \\
&\quad=\frac{i}{8m}\sum_{\alpha\neq\beta=1}^{N}\left(\Big[\big[p_{\alpha j},F_{\alpha\beta j}(|\mathbf{q}_\alpha-\mathbf{q}_\beta|)\big]_+,\delta(\mathbf{r}-\mathbf{q}_\alpha)\Big]_+ - \Big[\big[p_{\alpha j},F_{\alpha\beta j}(|\mathbf{q}_\alpha-\mathbf{q}_\beta|)\big]_+,\delta(\mathbf{r}-\mathbf{q}_\beta)\Big]_+\right) \\
&\quad=\frac{i}{8m}\sum_{\alpha\neq\beta=1}^{N}\Big[\big[p_{\alpha j},F_{\alpha\beta j}(|\mathbf{q}_\alpha-\mathbf{q}_\beta|)\big]_+,\big(\delta(\mathbf{r}-\mathbf{q}_\alpha)-\delta(\mathbf{r}-\mathbf{q}_\beta)\big)\Big]_+ \\
&\quad=\frac{i}{8m}\sum_{\alpha\neq\beta=1}^{N}\left[\big[p_{\alpha j},F_{\alpha\beta j}(|\mathbf{q}_\alpha-\mathbf{q}_\beta|)\big]_+,\left(-\frac{\partial}{\partial r_k}\mathcal{D}_k(\mathbf{r},\mathbf{q}_\alpha,\mathbf{q}_\beta)\right)\right]_+ \\
&\quad=-\frac{i}{8m}\frac{\partial}{\partial r_j}\sum_{\alpha\neq\beta=1}^{N}\Big[\big[p_{\alpha k},F_{\alpha\beta k}(|\mathbf{q}_\alpha-\mathbf{q}_\beta|)\big]_+,\mathcal{D}_j(\mathbf{r},\mathbf{q}_\alpha,\mathbf{q}_\beta)\Big]_+ ,
\end{aligned}
\tag{42}
$$

where use has been made of Eq. (28) in the next to last line and dummy indices r, k have been swapped in the last line. The energy equation is therefore

$$
\begin{aligned}
\frac{\partial}{\partial t}e(\mathbf{r},t) = &-\frac{1}{8m^2}\frac{\partial}{\partial r_j}\sum_{\alpha=1}^{N}\Big[p_\alpha^2,[p_{\alpha j},\delta(\mathbf{r}-\mathbf{q}_\alpha)]_+\Big]_+ \\
&+\frac{1}{4m}\sum_{\alpha=1}^{N}\Big[\big[p_{\alpha j},F_{\alpha j}^{\rm ext}(\mathbf{q}_\alpha)\big]_+,\delta(\mathbf{r}-\mathbf{q}_\alpha)\Big]_+ \\
&-\frac{1}{8m}\frac{\partial}{\partial r_j}\sum_{\alpha\neq\beta=1}^{N}\Big[\big[p_{\alpha k},F_{\alpha\beta k}(|\mathbf{q}_\alpha-\mathbf{q}_\beta|)\big]_+,\mathcal{D}_j(\mathbf{r},\mathbf{q}_\alpha,\mathbf{q}_\beta)\Big]_+ \\
&-\frac{\partial}{\partial r_j}\frac{1}{4m}\sum_{\alpha\neq\beta=1}^{N}\big[p_{\alpha j},U(|\mathbf{q}_\alpha-\mathbf{q}_\beta|)\delta(\mathbf{r}-\mathbf{q}_\alpha)\big]_+ .
\end{aligned}
\tag{43}
$$

Moving the three derivative terms to the left side, this can be rewritten as

$$
\frac{\partial}{\partial t}e(\mathbf{r},t)+\nabla\cdot s(\mathbf{r},t)=w(\mathbf{r},t)
\tag{44}
$$

with the energy flux

$$s_j(\mathbf{r},t) = \frac{1}{8m^2}\sum_{\alpha=1}^{N}\left[p_\alpha^2,[p_{\alpha j},\delta(\mathbf{r}-\mathbf{q}_\alpha)]_+\right]_+ + \frac{1}{8m}\sum_{\alpha\neq\beta=1}^{N}\left[\left[p_{\alpha k},F_{\alpha\beta k}(|\mathbf{q}_\alpha-\mathbf{q}_\beta|)\right]_+,\mathcal{D}_j(\mathbf{r},\mathbf{q}_\alpha,\mathbf{q}_\beta)\right]_+ + \frac{1}{4m}\sum_{\alpha\neq\beta=1}^{N}\left[p_{\alpha j},U(|\mathbf{q}_\alpha-\mathbf{q}_\beta|)\delta(\mathbf{r}-\mathbf{q}_\alpha)\right]_+ \tag{45}$$

and the work done by the external potential

$$w(\mathbf{r},t) = \frac{1}{4m}\sum_{\alpha=1}^{N}\left[\left[p_{\alpha j},F_{\alpha j}^{\mathrm{ext}}(\mathbf{q}_\alpha)\right]_+,\delta(\mathbf{r}-\mathbf{q}_\alpha)\right]_+ . \tag{46}$$

This work term can be simplified as follows,

$$\begin{aligned} w(\mathbf{r},t) &= \frac{1}{4m}\sum_\alpha\left(\mathbf{p}_\alpha\cdot\mathbf{F}^{\mathrm{ext}}(\mathbf{q}_\alpha,t)+\mathbf{F}^{\mathrm{ext}}(\mathbf{q}_\alpha,t)\cdot\mathbf{p}_\alpha\right)\delta(\mathbf{r}-\mathbf{q}_\alpha) \\ &\quad+\frac{1}{4m}\sum_\alpha\delta(\mathbf{r}-\mathbf{q}_\alpha)\left(\mathbf{p}_\alpha\cdot\mathbf{F}^{\mathrm{ext}}(\mathbf{q}_\alpha,t)+\mathbf{F}^{\mathrm{ext}}(\mathbf{q}_\alpha,t)\cdot\mathbf{p}_\alpha\right) \\ &= \frac{1}{4m}\sum_\alpha\left(2\mathbf{p}_\alpha,\cdot\mathbf{F}^{\mathrm{ext}}(\mathbf{r},t)\,\delta(\mathbf{r}-\mathbf{q}_\alpha)+\left[\mathbf{F}^{\mathrm{ext}}(\mathbf{q}_\alpha,t),\cdot\mathbf{p}_\alpha\right]\delta(\mathbf{r}-\mathbf{q}_\alpha)\right) \\ &\quad+\frac{1}{4m}\sum_\alpha\left(\delta(\mathbf{r}-\mathbf{q}_\alpha)\left[\mathbf{p}_\alpha,\cdot\mathbf{F}^{\mathrm{ext}}(\mathbf{q}_\alpha,t)\right]+2\delta(\mathbf{r}-\mathbf{q}_\alpha)\,\mathbf{F}^{\mathrm{ext}}(\mathbf{r},t)\cdot\mathbf{p}_\alpha\right) \\ &= \frac{1}{m}\mathbf{p}(\mathbf{r})\cdot\mathbf{F}^{\mathrm{ext}}(\mathbf{r},t)+\frac{1}{4m}\sum_\alpha\left[\mathbf{F}^{\mathrm{ext}}(\mathbf{q}_\alpha,t),\cdot\mathbf{p}_\alpha\right]\delta(\mathbf{r}-\mathbf{q}_\alpha) \\ &\quad+\frac{1}{4m}\sum_\alpha\delta(\mathbf{r}-\mathbf{q}_\alpha)\left[\mathbf{p}_\alpha,\cdot\mathbf{F}^{\mathrm{ext}}(\mathbf{q}_\alpha,t)\right] \\ &= \frac{1}{m}\mathbf{p}(\mathbf{r})\cdot\mathbf{F}^{\mathrm{ext}}(\mathbf{r},t)+\frac{i}{4m}\sum_\alpha\nabla_{\mathbf{q}_\alpha}\mathbf{F}^{\mathrm{ext}}(\mathbf{q}_\alpha,t)\,\delta(\mathbf{r}-\mathbf{q}_\alpha) \\ &\quad-\frac{i}{4m}\sum_\alpha\delta(\mathbf{r}-\mathbf{q}_\alpha)\,\nabla_{\mathbf{q}_\alpha}\mathbf{F}^{\mathrm{ext}}(\mathbf{q}_\alpha,t). \end{aligned}$$

This yields the final form for the work term as

$$w(\mathbf{r},t) = \frac{1}{m}\mathbf{p}(\mathbf{r})\cdot\mathbf{F}^{\mathrm{ext}}(\mathbf{r},t). \tag{47}$$

II. UNITARY TRANSFORMATION TO REST FRAME MOMENTA

Operators and phase functions of interest include those in the local reference frame. For simplicity consideration here is restricted to single particle functions,

$$A = A(\mathbf{q},\mathbf{p}-m\mathbf{u}(\mathbf{q})), \tag{48}$$

where $\mathbf{u}(\mathbf{q})$ is the average flow velocity at the particle position $\mathbf{q}$. In the classical case a change of variables in the momentum average to

$$\mathbf{p}' = \mathbf{p}-m\mathbf{u}(\mathbf{q}), \qquad \mathbf{q}' = \mathbf{q} \tag{49}$$

gives the simpler form

$$A' = A(\mathbf{q}',\mathbf{p}'), \tag{50}$$

which is independent of the velocity field. The objective here is to find a unitary operator that does the same for quantum mechanical operators,

$$A' = A(\mathbf{q}', \mathbf{p}') = UA(\mathbf{q}, \mathbf{p} - m\mathbf{u}(\mathbf{q}))U^{-1}. \tag{51}$$

The generator of the classical canonical transformation in Eq. (49) obeys[3]

$$p_i = \frac{\partial F(\mathbf{q}, \mathbf{p}')}{\partial q_i}, \qquad q_i' = \frac{\partial F(\mathbf{q}, \mathbf{p}')}{\partial p_i'}. \tag{52}$$

Using the identity generator $\mathbf{q} \cdot \mathbf{p}'$ a new generator for the deviation is introduced

$$F(\mathbf{q}, \mathbf{p}') = \mathbf{q} \cdot \mathbf{p}' + G(\mathbf{q}, \mathbf{p}'). \tag{53}$$

In the following $G(\mathbf{q}, \mathbf{p}')$ will be referred to as the generator rather than $F(\mathbf{q}, \mathbf{p}')$. Note that for this generator

$$p_i = p_i' + \frac{\partial G(\mathbf{q}, \mathbf{p}')}{\partial q_i}, \qquad q_i' = q_i + \frac{\partial G(\mathbf{q}, \mathbf{p}')}{\partial p_i'} \tag{54}$$

$$\frac{\partial G(\mathbf{q}, \mathbf{p}')}{\partial q_i} = mu_i(\mathbf{q}), \qquad \frac{\partial G(\mathbf{q}, \mathbf{p}')}{\partial p_i'} = 0 \tag{55}$$

so $G(\mathbf{q}, \mathbf{p}')$ is independent of $\mathbf{p}'$

$$G(\mathbf{q}, \mathbf{p}') = G(\mathbf{q}), \qquad \frac{\partial G(\mathbf{q})}{\partial q_i} = mw_i(\mathbf{q}). \tag{56}$$

Now consider the associated infinitesimal transformation, $mu_i(\mathbf{q}) \to \epsilon mu_i(\mathbf{q})$, $G(\mathbf{q}, \mathbf{p}') \to \epsilon G(\mathbf{q}, \mathbf{p}')$. To first order in ϵ,

$$\begin{aligned} A(\mathbf{q}', \mathbf{p}') - A(\mathbf{q}, \mathbf{p}) &\to \frac{\partial A(\mathbf{q}, \mathbf{p})}{\partial q_i}(q_i' - q_i) + \frac{\partial A(\mathbf{q}, \mathbf{p})}{\partial p_i}(p_i' - p_i) \\ &= \frac{\partial A(\mathbf{q}, \mathbf{p})}{\partial q_i}\epsilon\frac{\partial G(\mathbf{q})}{\partial p_i} - \frac{\partial A(\mathbf{q}, \mathbf{p})}{\partial p_i}\epsilon\frac{\partial G(\mathbf{q})}{\partial q_i} \\ &= \epsilon\{A, G\} \end{aligned} \tag{57}$$

where $\{,\}$ denotes the Poisson bracket.

The Dirac - Weyl quantization of this procedure is given by

$$\{A, G\} \to \frac{1}{i\hbar}[A, G] \tag{58}$$

although some limitations exist. According to this quantization procedure, the classical infinitesimal transformation corresponds to the finite transformation

$$A(\mathbf{q}', \mathbf{p}') = UA(\mathbf{q}, \mathbf{p})U^{-1}, \qquad U = e^{-\frac{1}{i\hbar}G}. \tag{59}$$

To demonstrate that this transformation does in fact give the desired result, transform a quantity $A(\mathbf{q}', \mathbf{r}')$,

$$A(\mathbf{q}', \mathbf{p}') = e^{-\frac{1}{i\hbar}G}A(\mathbf{q}, \mathbf{p})e^{\frac{1}{i\hbar}G} = A(e^{-\frac{1}{i\hbar}G}\mathbf{q}e^{\frac{1}{i\hbar}G}, e^{-\frac{1}{i\hbar}G}\mathbf{p}e^{\frac{1}{i\hbar}G}). \tag{60}$$

Since $G = G(\mathbf{q})$ it commutes with $\mathbf{q}$ and so

$$e^{-\frac{1}{i\hbar}G}\mathbf{q}e^{\frac{1}{i\hbar}G} = \mathbf{q}. \tag{61}$$

Next define

$$\mathbf{p}(\lambda) \equiv e^{-\frac{1}{i\hbar}\lambda G}\mathbf{p}e^{\frac{1}{i\hbar}\lambda G} \tag{62}$$

which obeys

$$\partial_\lambda \mathbf{p}(\lambda) = -e^{-\frac{1}{i\hbar}\lambda G}\frac{1}{i\hbar}\left[G(\mathbf{q}),\mathbf{p}\right]e^{\frac{1}{i\hbar}\lambda G} = -e^{-\frac{1}{i\hbar}\lambda G}\nabla_{\mathbf{q}}G(\mathbf{q})\,e^{\frac{1}{i\hbar}\lambda G}$$
$$= -m\mathbf{u}(\mathbf{q}). \tag{63}$$

The following identity was used above:

$$\left[f(\mathbf{q}),\mathbf{p}\right] = i\hbar\nabla_{\mathbf{q}}f(\mathbf{q}). \tag{64}$$

Integrating (63) from 0 to 1 gives

$$\mathbf{p}(1) = \mathbf{p}(0) - m\mathbf{u}(\mathbf{q}) \tag{65}$$

or

$$e^{-\frac{1}{i\hbar}G}\mathbf{p}e^{\frac{1}{i\hbar}G} = \mathbf{p} - m\mathbf{u}(\mathbf{q}). \tag{66}$$

Therefore (60) gives the desired result

$$A(\mathbf{q}',\mathbf{p}') = e^{-\frac{1}{i\hbar}G}A(\mathbf{q},\mathbf{p})e^{\frac{1}{i\hbar}G} = A(\mathbf{q},\mathbf{p}-m\mathbf{u}(\mathbf{q})). \tag{67}$$

In the text, most final results are expressed as local equilibrium correlation functions in the rest frame, e.g.

$$C_0(\mathbf{r},\mathbf{r}'\mid\mathbf{u}) \equiv \overline{A_0(\mathbf{r})\,B_0(\mathbf{r}')}^{\ell}. \tag{68}$$

Recall that the subscript zero denotes the chosen operator with all momenta replaced by their local rest frame momenta $\mathbf{p}_\alpha \to \mathbf{p}_\alpha - m\mathbf{u}(\mathbf{q}_\alpha)$. The local equilibrium ensemble also has this form. Then by cyclic invariance of the trace

$$C_0(\mathbf{r},\mathbf{r}'\mid\mathbf{u}) = \overline{e^{-\frac{1}{i\hbar}G}A_0(\mathbf{r})\,e^{\frac{1}{i\hbar}G}e^{-\frac{1}{i\hbar}G}B_0(\mathbf{r}')\,e^{-\frac{1}{i\hbar}G}}^{\ell}$$
$$= C_0(\mathbf{r},\mathbf{r}'\mid\mathbf{u}=\mathbf{0}) \tag{69}$$

and such correlation functions are independent of the local flow velocity.

III. TIME REVERSAL SYMMETRY

The objective of this section is to prove a property of local equilibrium correlation functions for operators that have a definite sign with respect to changes in the sign of the particle momentum operator. In the classical case this is accomplished by introduction of the parity operator for the momentum variables only. In the quantum case the parity operator changes both the momentum and position operators and the analysis no longer holds. Instead, an operator that changes the momentum sign but not the position operator sign is considered.

As in the classical case the quantum time-reversal operator T generates a reversal of motion operator in the sense

$$\mathcal{T}\mathbf{p}\mathcal{T}^{-1} = -\mathbf{p} \qquad\qquad \mathcal{T}\mathbf{q}\mathcal{T}^{-1} = \mathbf{q}. \tag{70}$$

It follows (e.g., coordinate representation for $\mathbf{p}$) that $\mathcal{T}i = -i\mathcal{T}$, and hence $\mathcal{T}$ is an anti-unitary operator

$$\mathcal{T}\left(c_1|\alpha\rangle + c_2|\beta\rangle\right) = c_1^*\mathcal{T}|\alpha\rangle + c_2^*\mathcal{T}|\beta\rangle, \tag{71}$$

where c_1 and c_2 are arbitrary complex numbers. In addition it is required that $\mathcal{T}$ be norm-preserving

$$\left|\left\langle\widetilde{\beta}|\widetilde{\alpha}\right\rangle\right| = |\langle\beta|\alpha\rangle| \tag{72}$$

where $|\alpha\rangle$ and $|\beta\rangle$ are members of a complete basis set $\{\alpha\}$, and

$$|\tilde{\alpha}\rangle = \mathcal{T}|\alpha\rangle, \qquad |\tilde{\beta}\rangle = \mathcal{T}|\beta\rangle. \tag{73}$$

It then follows that $\mathcal{T}$ is anti-unitary[4]

$$\left\langle\widetilde{\beta}|\widetilde{\alpha}\right\rangle = \langle\beta|\alpha\rangle^* . \tag{74}$$

Consider again the correlation function of (68)

$$C_0\left(\mathbf{r},\mathbf{r}' \mid \mathbf{u}=\mathbf{0}\right) \equiv \overline{A_0\left(\mathbf{r}\right) B_0\left(\mathbf{r}'\right)}^{\ell} . \tag{75}$$

It is shown here that this has the property

$$C_0\left(\mathbf{r},\mathbf{r}' \mid \mathbf{u}=\mathbf{0}\right) \equiv \overline{\left(\mathcal{T}A_0\left(\mathbf{r}\right)\mathcal{T}^{-1}\right)\left(\mathcal{T}B_0\left(\mathbf{r}'\right)\mathcal{T}^{-1}\right)}^{\ell} . \tag{76}$$

Thus if both $A_0\left(\mathbf{r}\right)$ and $B_0\left(\mathbf{r}'\right)$ have a definite sign under time reversal, the correlation function vanishes when the signs are different. The proof of (76) is complicated by the anti-unitary nature of $\mathcal{T}$ which does not obey the cyclic invariance of the trace.

The proof is as follows. Define

$$|\gamma\rangle = A^{\dagger}|\beta\rangle \xrightarrow{\text{dual correspondence}} \langle\gamma| = \langle\beta|A \tag{77}$$

where A is an arbitrary linear operator. Then

$$\langle\gamma|\alpha\rangle = \langle\alpha|\gamma\rangle^* = \langle\widetilde{\alpha}|\widetilde{\gamma}\rangle \tag{78}$$

where the second equality follows from the anti-unitarity of $\mathcal{T}$, (74). Writing $\langle\gamma|\alpha\rangle$ and $\langle\widetilde{\alpha}|\widetilde{\gamma}\rangle$ explictly in terms of A

$$\langle\beta|\, A\,|\alpha\rangle = \langle\widetilde{\alpha}|\,\mathcal{T}A^{\dagger}\,|\beta\rangle = \langle\widetilde{\alpha}|\,\mathcal{T}A^{\dagger}\mathcal{T}^{-1}\left|\widetilde{\beta}\right\rangle . \tag{79}$$

As a special case, the diagonal elements for a self-adjoint operator are

$$\langle\alpha|\, A\,|\alpha\rangle = \langle\widetilde{\alpha}|\,\mathcal{T}A\mathcal{T}^{-1}\,|\widetilde{\alpha}\rangle . \tag{80}$$

Now the correlation function in (75) can be expressed as

$$C_0\left(\mathbf{r},\mathbf{r}' \mid \mathbf{u}=\mathbf{0}\right) \equiv \sum_N \mathrm{Tr} A_0\left(\mathbf{r}\right) B_0\left(\mathbf{r}'\right)\rho^{\ell} \tag{81}$$

$$= \sum_N \sum_\alpha \langle\alpha|\, A_0\left(\mathbf{r}\right) B_0\left(\mathbf{r}'\right)\rho^{\ell}\,|\alpha\rangle \tag{82}$$

$$= \sum_N \sum_\alpha \langle\widetilde{\alpha}|\,\mathcal{T}A_0\left(\mathbf{r}\right) B_0\left(\mathbf{r}'\right)\rho^{\ell}\mathcal{T}^{-1}\,|\widetilde{\alpha}\rangle . \tag{83}$$

The last line follows from (80). Since $|\alpha\rangle$ and $|\widetilde{\alpha}\rangle$ are in one-to-one correspondence the latter is a complete basis set. Furthermore

$$\mathcal{T}\rho^{\ell}\mathcal{T}^{-1} = \rho^{\ell}$$

and (83) becomes (76).

IV. TRANSFORMATION TO LOCAL REST FRAME

Operator functions of the particle positions and momenta typically refer to a fixed laboratory frame. They define both convective motion and motion relative to each particle's local average velocity field. Denote such an operator by $A(\{\mathbf{q},\mathbf{p}\})$, where $\{\mathbf{q},\mathbf{p}\}$ denotes the N particle positions and momenta. It is convenient to extract from this the corresponding operator representing only motion relative to the average flow $A_0(\{\mathbf{q},\mathbf{p}\}) \equiv A(\{\mathbf{q},\mathbf{p}-m\mathbf{u}\left(\mathbf{q}\right)\})$. They are related by the generator of Eq. (67),

$$A_0(\{\mathbf{q},\mathbf{p}\}) = e^{-\frac{1}{i\hbar}G} A(\{\mathbf{q},\mathbf{p}\}) e^{\frac{1}{i\hbar}G} . \tag{84}$$

Operators of primary interest are the conserved densities and associated fluxes,

$$\{\psi(\mathbf{r})\} = \{n(\mathbf{r}), e(\mathbf{r}), \mathbf{p}(\mathbf{r})\}, \tag{85}$$

$$\{\gamma_i\} = \{p_i, s_i, t_{ij}\}\,. \tag{86}$$

The corresponding rest frame operators are easily found to be

$$\{\psi_0(\mathbf{r})\} = \begin{pmatrix} n(\mathbf{r}) \\ e(\mathbf{r}) - \mathbf{u}(\mathbf{r})\cdot\mathbf{p}(\mathbf{r}) + \frac{1}{2}u^2 mn(\mathbf{r}) \\ p_x(\mathbf{r}) - mn(\mathbf{r})u_x(\mathbf{r}) \\ p_y(\mathbf{r}) - mn(\mathbf{r})u_y(\mathbf{r}) \\ p_z(\mathbf{r}) - mn(\mathbf{r})u_z(\mathbf{r}) \end{pmatrix} \tag{87}$$

and the rest frame fluxes are

$$\{\gamma_{0i}(\mathbf{r})\} = \begin{pmatrix} p_i(\mathbf{r}) \\ s_i(\mathbf{r}) - u_j(\mathbf{r})t_{ji}(\mathbf{r}) + \frac{u^2}{2}p_i(\mathbf{r}) \\ t^T_{ix}(\mathbf{r}) - p_i(\mathbf{r})u_x(\mathbf{r}) \\ t^T_{iy}(\mathbf{r}) - p_i(\mathbf{r})u_y(\mathbf{r}) \\ t^T_{iz}(\mathbf{r}) - p_i(\mathbf{r})u_z(\mathbf{r}) \end{pmatrix} - u_i(\mathbf{r}) \begin{pmatrix} n(\mathbf{r}) \\ e(\mathbf{r}) - \mathbf{p}(\mathbf{r})\cdot\mathbf{u}(\mathbf{r}) + \frac{1}{2}mn(\mathbf{r})u^2(\mathbf{r}) \\ p_x(\mathbf{r}) - mn(\mathbf{r})u_x(\mathbf{r}) \\ p_y(\mathbf{r}) - mn(\mathbf{r})u_y(\mathbf{r}) \\ p_z(\mathbf{r}) - mn(\mathbf{r})u_z(\mathbf{r}) \end{pmatrix}. \tag{88}$$

A concise form for these transformations to the local rest frame can be described in matrix representation,

$$\psi_{0\alpha}(\mathbf{r}) = A_{\alpha\beta}(\mathbf{u})\psi_\beta(\mathbf{r})\,, \tag{89}$$

and the flux vector transforms to the rest frame as

$$\gamma_{0i\alpha}(\mathbf{r}) = A_{\alpha\beta}(\mathbf{u})\left(\gamma_{i\beta} - u_i(\mathbf{r})\,\psi_\beta(\mathbf{r})\right) \tag{90}$$

where the matrix $A(\mathbf{u})$ is given by

$$A(\mathbf{u}) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{1}{2}mu^2 & 1 & -u_x & -u_y & -u_z \\ -mu_x & 0 & 1 & 0 & 0 \\ -mu_y & 0 & 0 & 1 & 0 \\ -mu_z & 0 & 0 & 0 & 1 \end{pmatrix} \tag{91}$$

$$\equiv \begin{pmatrix} 1 & 0 & \mathbf{0}^T \\ \frac{1}{2}mu^2 & 1 & -\mathbf{u}^T \\ -m\mathbf{u} & \mathbf{0} & \mathbf{I} \end{pmatrix}. \tag{92}$$

The second equality is a simplified notation, whereby $\mathbf{u}$ is a three-component column vector, $\mathbf{u}^T$ is its transpose, $\mathbf{0}^T$ is the transposed zero vector, and $\mathbf{I}$ is the 3×3 identity matrix. To obtain (90) use has been made of the fact that

$$A^{-1}(\mathbf{u}) = A(-\mathbf{u}). \tag{93}$$

It is clear that the results (89) and (90) apply as well for the average conserved densities and fluxes

$$\overline{\psi}_{0\alpha}(\mathbf{r}) = A_{\alpha\beta}(\mathbf{u})\overline{\psi}_\beta(\mathbf{r}) \tag{94}$$

and the flux vector transforms to the rest frame as

$$\overline{\gamma}_{0i\alpha}(\mathbf{r}) = A_{\alpha\beta}(\mathbf{u})\left(\overline{\gamma}_{i\beta} - u_i(\mathbf{r})\,\overline{\psi}_\beta(\mathbf{r})\right) \tag{95}$$

since the average is a linear operation and commutes with A.

As an example, consider the local equilibrium ensemble given by (see Eq. (22) in the main text)

$$\rho^\ell_N\left[y(t)\right] = e^{-Q[y(t)] - \int d\mathbf{r} y_\alpha(r,t)\psi_\alpha(\mathbf{r})} \tag{96}$$

where the conjugate fields $\{y(\mathbf{r},t)\}$ are

$$\{y(\mathbf{r},t)\}=\left\{\left(-\nu(\mathbf{r},t)+\frac{1}{2}\beta(\mathbf{r},t)mu^2(\mathbf{r},t)\right),\beta(\mathbf{r},t),-\beta(\mathbf{r},t)\mathbf{u}(\mathbf{r},t)\right\}. \tag{97}$$

The local rest frame form for (96) is obtained directly from (89)

$$\begin{aligned}\int dry_\alpha\left(r,t\right)\psi_\alpha\left(\mathbf{r}\right)&=\int dry_\alpha\left(r,t\right)A_{\alpha\beta}(\mathbf{u})\psi_{0\beta}\left(\mathbf{r}\right)\\&=\int drA^T_{\beta\alpha}(\mathbf{u})y_\alpha\left(r,t\right)\psi_{0\beta}\left(\mathbf{r}\right)\\&=\int dry_{0\alpha}\left(r,t\right)\psi_{0\alpha}\left(\mathbf{r}\right).\end{aligned} \tag{98}$$

Here $\{y_0\left(\mathbf{r},t\right)\}$ are the rest frame conjugate variables

$$y_{0\alpha}(\mathbf{r})=A^T_{\alpha\beta}y_\beta(\mathbf{r})=\{-\nu(\mathbf{r},t),\beta(\mathbf{r},t),0\} \tag{99}$$

where $y_0(\mathbf{r},t)=y(\mathbf{r},t)|_{u=0}$ and A^T is the transpose of A. The local equilibrium ensemble is therefore expressed in terms of the rest frame variables

$$\rho^\ell\left[y\left(t\right)\right]=\rho_0^\ell\left[y_0\left(t\right)\right]=e^{-Q[y_0(t)]-\int dry_{0\alpha}(r,t)\psi_{0\alpha}(\mathbf{r})}. \tag{100}$$

As a second example consider the correlations for the derivatives of the average conserved fields with respect to the conjugate fields (see Eq. (115) in the main text),

$$g_{\alpha\beta}\left(\mathbf{r},\mathbf{r}'|y(t)\right)=\frac{\delta^2Q(t)}{\delta y_\alpha\left(\mathbf{r}\right)\delta y_\beta\left(\mathbf{r}'\right)}=\frac{\delta\overline{\psi}^\ell_\beta\left(\mathbf{r}'\right)}{\delta y_\alpha\left(\mathbf{r}\right)}=\frac{\delta\overline{\psi}^\ell_\alpha\left(\mathbf{r}\right)}{\delta y_\beta\left(\mathbf{r}'\right)} \tag{101}$$

$$=\left.\overline{\psi_\alpha\left(\mathbf{r}\right)\widetilde{\psi}_\beta\left(\mathbf{r}',t\right)}^\ell\right|_{y(t)}, \tag{102}$$

with the tilde representing a transform closely related to the Kubo transform for correlation functions,

$$\widetilde{\psi}_\alpha(\mathbf{r}|y(t))=\int_0^1e^{-z\eta[y(t)]}\left(\psi_\alpha(\mathbf{r})-\overline{\psi}^\ell_\alpha\left(\mathbf{r}|y(t)\right)\right)e^{z\eta[y(t)]}dz \tag{103}$$

and the local equilibrium average is defined by

$$\left.\overline{X(\mathbf{r})Y(\mathbf{r}')}^\ell\right|_{y(t)}\equiv\left\langle X(\mathbf{r})Y(\mathbf{r}');\rho^\ell[y(t)]\right\rangle. \tag{104}$$

It is readily seen that

$$\widetilde{\psi}_{0\alpha}(\mathbf{r}|y(t))=A_{\alpha\beta}(\mathbf{u}\left(\mathbf{r}\right))\widetilde{\psi}_\beta(\mathbf{r}|y(t)). \tag{105}$$

Therefore, in matrix notation, $g\left(\mathbf{r},\mathbf{r}'|y(t)\right)$ transforms as

$$g\left(\mathbf{r},\mathbf{r}'\mid y(t)\right)=A^{-1}\left(\mathbf{u}\left(\mathbf{r}\right)\right)g_0\left(\mathbf{r},\mathbf{r}'\mid y(t)\right)A^{T-1}\left(\mathbf{u}\left(\mathbf{r}'\right)\right) \tag{106}$$

with

$$g_0\left(\mathbf{r},\mathbf{r}'\mid y_0\right)=\left.\overline{\psi_{0\alpha}\left(\mathbf{r}\right)\widetilde{\psi}_{0\beta}\left(\mathbf{r}',t\right)}^\ell\right|_{y(t)}. \tag{107}$$

Since $g_0\left(\mathbf{r},\mathbf{r}'\mid y_0\right)$ is now in the rest frame, the unitary transformation (69) can be used to eliminate the dependence on $\mathbf{u}\left(\mathbf{r}\right)$. Next, the time-reversal symmetry (76) implies that matrix elements of pairs that transform oppositely under momentum reversal must vanish. Consider the densities $\psi_\alpha(\mathbf{r})$. For $\alpha=\{1,2\}$ the density is unchanged under time reversal, while the densities corresponding to $\alpha = \{3,4,5\}$ change sign. The local equilibrium ensemble index function $\eta[y_0(t)]$ is also unchanged. Therefore, $g_0(\mathbf{r},\mathbf{r}' \mid y_0)$ has the form

$$g_0(\mathbf{r},\mathbf{r}' \mid y(t)) = \begin{pmatrix} \frac{\delta\overline{n}^{\ell}(\mathbf{r})}{\delta\nu(\mathbf{r}')}\Big|_{\beta} & -\frac{\delta\overline{n}^{\ell}(\mathbf{r})}{\delta\beta(\mathbf{r}')}\Big|_{\nu} & \mathbf{0}^T \\ \frac{\delta\overline{e_0}^{\ell}(\mathbf{r})}{\delta\nu(\mathbf{r}')}\Big|_{\beta} & -\frac{\delta\overline{e_0}^{\ell}(\mathbf{r})}{\delta\beta(\mathbf{r}')}\Big|_{\nu} & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & \frac{n(\mathbf{r})}{\beta(\mathbf{r})}\delta(\mathbf{r}-\mathbf{r}')\mathbf{I} \end{pmatrix} \tag{108}$$

where use has been made of (101) in the local rest frame, and for $\alpha,\beta = 3,4,5$

$$g_{0\alpha\beta}(\mathbf{r},\mathbf{r}' \mid y(t)) = \delta_{\alpha\beta}\frac{1}{m\beta(\mathbf{r})}\frac{\delta\overline{p}_x^L(\mathbf{r})}{\delta u_x(\mathbf{r}')}\,|_{\mathbf{u}=0} = \delta_{\alpha\beta}\frac{mn(\mathbf{r})}{m\beta(\mathbf{r}')} = \frac{n(\mathbf{r})}{\beta(\mathbf{r})}\delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}'). \tag{109}$$

The inverse matrix $g_{0\alpha\beta}^{-1}(\mathbf{r},\mathbf{r}')$, defined by

$$\int d\mathbf{r}'' g_{0\alpha\gamma}(\mathbf{r},\mathbf{r}'')g_{0\gamma\beta}^{-1}(\mathbf{r}'',\mathbf{r}') = \delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}'), \tag{110}$$

is required below as well, and is found to be

$$g_0^{-1}(\mathbf{r},\mathbf{r}' \mid y(t)) = \begin{pmatrix} \frac{\delta\nu(\mathbf{r})}{\delta\overline{n}(\mathbf{r}')}\,|_{e_0} & \frac{\delta\nu(\mathbf{r})}{\delta\overline{e_0}(\mathbf{r}')}\,|_{n} & \mathbf{0}^T \\ -\frac{\delta\beta(\mathbf{r})}{\delta\overline{n}(\mathbf{r}')}\,|_{e_0} & -\frac{\delta\beta(\mathbf{r})}{\delta\overline{e_0}(\mathbf{r}')}\,|_{n} & \mathbf{0}^T \\ \mathbf{0} & \mathbf{0} & \frac{\beta(\mathbf{r})}{n(\mathbf{r}')}\delta(\mathbf{r}-\mathbf{r}')\mathbf{I} \end{pmatrix}. \tag{111}$$

V. LIOUVILLE EQUATION SOLUTION $\Delta(t)$ IN THE REST FRAME.

The formal solution to the Liouville equation for the deviation from local equilibrium is given in Appendix C of the text

$$\Delta(t) = \int_0^t dt' e^{-\mathcal{L}(t-t')}\int d\mathbf{r}\left(\Psi_\alpha(\mathbf{r}|y(t'))\frac{\partial}{\partial r_i}\overline{\gamma}_{i\alpha}^*(\mathbf{r},t') - \Phi_{i\alpha}(\mathbf{r}|y(t'))\frac{\partial}{\partial r_i}\overline{\psi}_\alpha(\mathbf{r},t')\right)\rho_N^{\ell}[y(t')], \tag{112}$$

$$\Phi_{i\alpha}(\mathbf{r}|y(t)) = \Gamma_{i\alpha}(\mathbf{r}|y(t)) - \int d\mathbf{r}'\Psi_\beta(\mathbf{r}'|y(t))\,\overline{\psi_\beta(\mathbf{r}'))\Gamma_{i\alpha}(\mathbf{r}|y(t))}^{\ell}\Big|_{y(t)}, \tag{113}$$

$$\Psi_\alpha(\mathbf{r}|y(t)) = \int d\mathbf{r}'\widetilde{\psi}_\beta(\mathbf{r}'|y(t))g_{\beta\alpha}^{-1}(\mathbf{r}',\mathbf{r} \mid y(t)), \tag{114}$$

$$\Gamma_{i\alpha}(\mathbf{r}|y(t)) = \int d\mathbf{r}'\widetilde{\gamma}_{i\beta}(\mathbf{r}'|y(t))g_{\beta\alpha}^{-1}(\mathbf{r}',\mathbf{r} \mid y(t)). \tag{115}$$

Since $\Psi_\alpha(\mathbf{r}|y(t))$ and $\Gamma_{i\alpha}(\mathbf{r}|y(t))$ are simply related to the local conserved density $\psi_\alpha(\mathbf{r})$ and their fluxes $\gamma_{i\alpha}(\mathbf{r})$, the transformation of the latter follow directly from the former (recall the transformations of $\widetilde{\psi}(\mathbf{r}|y(t))$ and $\widetilde{\gamma}(\mathbf{r}|y(t))$ are the same as those of $\psi(\mathbf{r})$ and $\gamma(\mathbf{r})$, see (105)).

Consider first $\Psi_\alpha(\mathbf{r}|y(t))$

$$\begin{aligned}\Psi_\alpha(\mathbf{r}|y(t)) &= \int d\mathbf{r}' A_{\alpha\beta}^{-1}(\mathbf{u}(\mathbf{r}'))\,\widetilde{\psi}_{0\beta}(\mathbf{r}'|y(t))\left(A^T(\mathbf{u}(\mathbf{r}'))\,g_0^{-1}(\mathbf{r}',\mathbf{r} \mid y(t))\,A(\mathbf{u}(\mathbf{r}))\right)_{\beta\alpha} \\ &= \int d\mathbf{r}'\widetilde{\psi}_{0\beta}(\mathbf{r}'|y(t))\left(g_0^{-1}(\mathbf{r}',\mathbf{r} \mid y(t))\,A(\mathbf{u}(\mathbf{r}))\right)_{\beta\alpha} \\ &= \Psi_{0\gamma}(\mathbf{r}|y(t))A_{\gamma\alpha}(\mathbf{u}(\mathbf{r})).\end{aligned} \tag{116}$$

In a similar way, using (95), $\Gamma_{i\alpha}(\mathbf{r}|y(t))$ becomes

$$\begin{aligned}\Gamma_{i\alpha}(\mathbf{r}|y(t)) &= \int d\mathbf{r}' A_{\beta\sigma}^{-1}(\mathbf{u})\left(\widetilde{\gamma}_{0\sigma}(\mathbf{r}'|y(t)) + u_i(\mathbf{r}')\widetilde{\psi}_{0\sigma}(\mathbf{r}'|y(t))\right)\left(A^T(\mathbf{u}(\mathbf{r}'))\,g_0^{-1}(\mathbf{r}',\mathbf{r} \mid y(t))\,A(\mathbf{u}(\mathbf{r}))\right)_{\beta\alpha} \\ &= \int d\mathbf{r}'\left(\widetilde{\gamma}_{0\sigma}(\mathbf{r}'|y(t)) + u_i(\mathbf{r}')\widetilde{\psi}_{0\sigma}(\mathbf{r}'|y(t))\right)\left(g_0^{-1}(\mathbf{r}',\mathbf{r} \mid y(t))\,A(\mathbf{u}(\mathbf{r}))\right)_{\sigma\alpha} \\ &= \Gamma_{i0\beta}(\mathbf{r}|y(t))A_{\beta\alpha}(\mathbf{u}(\mathbf{r})) + \int d\mathbf{r}' u_i(\mathbf{r}')\widetilde{\psi}_{0\sigma}(\mathbf{r}'|y(t))g_{0\sigma\nu}^{-1}(\mathbf{r}',\mathbf{r} \mid y(t))\,A_{\nu\alpha}(\mathbf{u}(\mathbf{r})).\end{aligned} \tag{117}$$

Finally, with these results the transformation of $\Phi_i(\mathbf{r}|y(t))$ is obtained,

$$\begin{aligned}
\Phi_{i\alpha}(\mathbf{r}|y(t)) =&\Gamma_{i\alpha}(\mathbf{r}|y(t)) - \int d\mathbf{r}'\ \Psi_\beta(\mathbf{r}')\ \overline{\psi_\beta(\mathbf{r}')\Gamma_{i\alpha}(\mathbf{r}|y(t))}^{\ell}\Big|_{y(t)} \\
=&\Gamma_{i0\beta}(\mathbf{r}|y(t))A_{\beta\alpha}\left(\mathbf{u}\left(\mathbf{r}\right)\right) + \int d\mathbf{r}'\ \left(u_i(\mathbf{r}')\psi_{0\sigma}(\mathbf{r}')\right) g^{-1}_{0\sigma\nu}\left(\mathbf{r}', \mathbf{r} \mid y(t)\right) A_{\nu\alpha}\left(\mathbf{u}\left(\mathbf{r}\right)\right) \\
&- \int d\mathbf{r}'\ \Psi_{0\sigma}(\mathbf{r}'|y(t))\ \overline{\psi_{0\sigma}(\mathbf{r}')\Gamma_{0i\beta}(\mathbf{r}|y(t))}^{\ell}\Big|_{y(t)} A_{\beta\alpha}\left(\mathbf{u}\left(\mathbf{r}\right)\right) \\
&- \int d\mathbf{r}'\ \Psi_{0\sigma}(\mathbf{r}'|y(t)) \int d\mathbf{r}'' u_i(\mathbf{r}'')\ \overline{\psi_{0\beta}(\mathbf{r}')\widetilde{\psi}_{0\sigma}(\mathbf{r}''|y(t))}^{\ell}\Big|_{y(t)} g^{-1}_{0\sigma\nu}\left(\mathbf{r}'', \mathbf{r} \mid y(0)\right) A_{\nu\alpha}\left(\mathbf{u}\left(\mathbf{r}\right)\right)^{\ell}(t) \\
=& \left(\Gamma_{i0\beta}(\mathbf{r}|y(t)) - \int d\mathbf{r}'\ \Psi_{0\sigma}(\mathbf{r}'|y(t))\ \overline{\psi_{0\sigma}(\mathbf{r}')\Gamma_{0i}(\mathbf{r}|y(t))}^{\ell}\Big|_{y(t)}\right) A_{\beta\alpha}\left(\mathbf{u}\left(\mathbf{r}\right)\right),
\end{aligned} \tag{118}$$

$$\Phi_{i\alpha}(\mathbf{r}|y) = \Phi_{0i\beta}(\mathbf{r}|y)A_{\beta\alpha}\left(\mathbf{u}\left(\mathbf{r}\right)\right). \tag{119}$$

The solution to the Liouville equation (112) is therefore

$$\Delta\left(t\right) = \int_0^t dt' e^{-\mathcal{L}(t-t')} \int d\mathbf{r} \left(\Psi_{0\beta}(\mathbf{r}|y(t'))A_{\beta\alpha}\left(\mathbf{u}\left(\mathbf{r}\right)\right) \frac{\partial}{\partial r_i}\overline{\gamma}^*_{i\alpha}\left(\mathbf{r}, t'|y\right) - \Phi_{i0\beta}(\mathbf{r}|y(t'))A_{\beta\alpha}\left(\mathbf{u}\left(\mathbf{r}\right)\right) \frac{\partial}{\partial r_i}\overline{\psi}_\alpha\left(\mathbf{r}, t'\right)\right) \rho^{\ell}_N[y\left(t'\right)]. \tag{120}$$

The operators $\Psi_{0\beta}(\mathbf{r}|y(t'))$ and $\Phi_{i0\beta}(\mathbf{r}|y(t'))$ with subscript 0 are the same as those in (112) except with $\mathbf{u}\left(\mathbf{r}\right) = 0$.

It remains to transform $\partial_i\overline{\gamma}^*_{i\alpha}\left(\mathbf{r}, t'\right)$ and $\partial_i\overline{\psi}_\alpha\left(\mathbf{r}, t\right)$ to the corresponding rest frame form. First, note that $\Psi_{0\beta}(\mathbf{r}|y(t'))$ and $\Phi_{0i\beta}(\mathbf{r}|y(t'))$ can be written as

$$\Psi_{0\alpha}(\mathbf{r}|y(t)) = \int d\mathbf{r}'\widetilde{\psi}_{0\beta}(\mathbf{r}'|y(t))g^{-1}_{0\beta\alpha}\left(\mathbf{r}', \mathbf{r} \mid y(t)\right), \tag{121}$$

$$\Phi_{0i\beta}(\mathbf{r}|y(t)) = \int d\mathbf{r}'\widetilde{\phi}_{0i\beta}(\mathbf{r}'|y(t))g^{-1}_{0\beta\alpha}\left(\mathbf{r}', \mathbf{r} \mid y(t)\right) \tag{122}$$

where

$$\widetilde{\phi}_{0i\beta}(\mathbf{r}|y(t)) = \widetilde{\gamma}_{i0\beta}(\mathbf{r}|y(t)) - \int d\mathbf{r}' \int d\mathbf{r}''\ \widetilde{\psi}_{0\sigma}(\mathbf{r}'|y(t))g^{-1}_{0\sigma\alpha}\left(\mathbf{r}', \mathbf{r}'' \mid y(t)\right) \overline{\psi_{0\sigma}(\mathbf{r}'')\widetilde{\gamma}_{0i\beta}(\mathbf{r}|y(t))}^{\ell}\Big|_{y(t)}. \tag{123}$$

Then

$$\begin{aligned}
\int d\mathbf{r}\Phi_{i0\beta}(\mathbf{r}|y(t'))A_{\beta\alpha}\left(\mathbf{u}\left(\mathbf{r}\right)\right)\partial_i\overline{\psi}_\alpha\left(\mathbf{r}, t\right) &= \int d\mathbf{r} \int d\mathbf{r}'\widetilde{\phi}_{0i\beta}(\mathbf{r}'|y(t))g^{-1}_{0\beta\alpha}\left(\mathbf{r}', \mathbf{r} \mid y_0\right) A_{\alpha\nu}\left(\mathbf{u}\left(\mathbf{r}\right)\right) \frac{\partial}{\partial r_i}\overline{\psi}_\nu\left(\mathbf{r}, t\right) \\
&= \int d\mathbf{r} \int d\mathbf{r}'\widetilde{\phi}_{0i\beta}(\mathbf{r}'|y(t))A^{T-1}_{\beta\alpha}(\mathbf{r}') \int d\mathbf{r} g^{-1}_{\alpha\sigma}(\mathbf{r}', \mathbf{r} \mid y_0)\frac{\partial}{\partial r_i}\overline{\psi}_\sigma\left(\mathbf{r}, t\right) \\
&= - \int d\mathbf{r}'\widetilde{\phi}_{0i\beta}(\mathbf{r}'|y(t))A^{T-1}_{\beta\alpha}(\mathbf{r}')\frac{\partial}{\partial r'_i}y_\alpha\left(\mathbf{r}', t\right) \\
&= - \int d\mathbf{r}'\widetilde{\phi}_{0i\beta}(\mathbf{r}'|y(t))A^{T-1}_{\beta\alpha}(\mathbf{u})\frac{\partial}{\partial r'_i}A^{T}_{\alpha\sigma}(\mathbf{u})y_{0\sigma}\left(\mathbf{r}', t\right).
\end{aligned} \tag{124}$$

Finally, then

$$\int d\mathbf{r}\Phi_{i0\beta}(\mathbf{r}|y(t'))A_{\beta\alpha}\left(\mathbf{u}\left(\mathbf{r}\right)\right)\frac{\partial}{\partial r_i}\overline{\psi}_\alpha\left(\mathbf{r}, t\right) = \int d\mathbf{r}' \left(-\widetilde{\phi}_{0i2}(\mathbf{r}'|y(t))\frac{\partial}{\partial r'_i}\beta(\mathbf{r}', t') + \widetilde{\phi}_{0ij}(\mathbf{r}'|y(t))\beta(\mathbf{r}', t')\frac{\partial}{\partial r'_j}u_i(\mathbf{r}', t')\right). \tag{125}$$

In a similar way

$$\begin{aligned}
\Psi_{0\beta}(\mathbf{r}|y(t))A_{\beta\alpha}\left(\mathbf{u}\left(\mathbf{r}\right)\right)\frac{\partial}{\partial r_i}\overline{\gamma}^*_{i\alpha}\left(\mathbf{r}, t|\, y\right) &= \Psi_{0\beta}(\mathbf{r}|y(t'))A_{\beta\alpha}\left(\mathbf{u}\left(\mathbf{r}\right)\right)\frac{\partial}{\partial r_i}\overline{\gamma}^*_{i\alpha}\left(\mathbf{r}, t'|y\right) \\
&= \Psi_{02}(\mathbf{r}|y(t))\left(\nabla\cdot\overline{\mathbf{s}}^*_0(\mathbf{r}, t|y) + \overline{t}^*_{ij}\left(\mathbf{r}, t'|y\right)\partial_i u_j\right) + \Psi_{0j}(\mathbf{r}|y(t))\overline{t}^*_{ij}(\mathbf{r}, t|y).
\end{aligned} \tag{126}$$

In the above solution to the Liouville equation $\overline{\gamma}_i^*\left(\mathbf{r},t'|y\right)$ are the laboratory frame fluxes, $\overline{\gamma}_i^*\left(\mathbf{r},t'|y\right)=\left\langle \gamma_i\left(\mathbf{r}\right);t\right\rangle -\left\langle \gamma_i\left(\mathbf{r}\right);t\right\rangle^{\ell}$. The corresponding rest frame fluxes $\overline{\gamma}_{0i}^*\left(\mathbf{r},t'|y\right)=\left\langle \gamma_{0i}\left(\mathbf{r}\right);t\right\rangle -\left\langle \gamma_{0i}\left(\mathbf{r}\right);t\right\rangle^{\ell}$ differ for the energy flux $\overline{\gamma}_{02i}^*\left(\mathbf{r},t'|y\right)=\overline{\gamma}_{2i}^*\left(\mathbf{r},t'|y\right)-u_j\left(\mathbf{r},t'\right)\overline{t}_{ij}^*\left(\mathbf{r},t'|y\right)\equiv\overline{s}_{0i}^*(\mathbf{r},t'|y)$.

The form for the solution to the Liouville equation (120) becomes

$$\Delta\left(t\right)=\int_0^t dt' e^{-\mathcal{L}\left(t-t'\right)}\int d\mathbf{r}\left(\widetilde{\phi}_{0i2}(\mathbf{r}|y(t))\frac{\partial}{\partial r_i}\beta(\mathbf{r},t')-\widetilde{\sigma}_{0ij}(\mathbf{r}|y(t'))\beta(\mathbf{r},t')\frac{\partial}{\partial r_j}u_i(\mathbf{r},t')\right.$$
$$\left.+\Psi_{02}(\mathbf{r}|y(t'))\nabla\cdot\overline{\mathbf{s}}_0^*(\mathbf{r},t'|y)+\Psi_{0j}(\mathbf{r}|y(t'))\partial_i\overline{t}_{ij}^*(\mathbf{r},t'|y)\right)\rho_N^{\ell}[y\left(t'\right)]. \tag{127}$$

More explicitly

$$\widetilde{\phi}_{0i2}(\mathbf{r}|y(t))=\widetilde{s}_{0i}(\mathbf{r}|y(t))-\int d\mathbf{r}'\int d\mathbf{r}''\;\widetilde{\psi}_{0\sigma}(\mathbf{r}'|y(t))g_{0\sigma\alpha}^{-1}\left(\mathbf{r}',\mathbf{r}''\mid y(t)\right)\left.\overline{\psi_{0\sigma}(\mathbf{r}'')\widetilde{s}_{0i}(\mathbf{r}|y(t))}^{\ell}\right|_{y(t)}. \tag{128}$$

By time reversal symmetry $\left.\overline{\psi_{0\sigma}(\mathbf{r}'')\widetilde{s}_{0i}(\mathbf{r}|y(t))}^{\ell}\right|_{y(t)}$ is non-vanishing only for $\psi_{0\sigma}(\mathbf{r}'')=p_i(\mathbf{r}'')$, and similarly $g_{0\sigma\alpha}^{-1}\left(\mathbf{r}',\mathbf{r}''\mid y(t)\right)$ is non-zero only for the diagonal momentum components, so

$$\begin{aligned}\widetilde{\phi}_{0i2}(\mathbf{r}|y(t))&=\widetilde{s}_{0i}(\mathbf{r}|y(t))-\int d\mathbf{r}'\int d\mathbf{r}''\;\widetilde{p}_{0j}(\mathbf{r}'|y(t))\delta\left(\mathbf{r}'-\mathbf{r}''\right)\frac{\beta(\mathbf{r}',t')}{mn(\mathbf{r}',t')}\left.\overline{p_{0j}(\mathbf{r}'')\widetilde{s}_{0i}(\mathbf{r}|y(t))}^{\ell}\right|_{y(t)}\\&=\widetilde{s}_{0i}(\mathbf{r}|y(t))-\int d\mathbf{r}'\;\widetilde{p}_{0j}(\mathbf{r}'|y(t))\frac{\beta(\mathbf{r}',t')}{mn(\mathbf{r}',t')}\left.\overline{p_{0j}(\mathbf{r}')\widetilde{s}_{0i}(\mathbf{r}|y(t))}^{\ell}\right|_{y(t)}.\end{aligned} \tag{129}$$

Also

$$\begin{aligned}\Psi_{02}(\mathbf{r}|y(t))&=\int d\mathbf{r}'\widetilde{\psi}_{0\beta}(\mathbf{r}'|y(t))g_{0\beta2}^{-1}\left(\mathbf{r}',\mathbf{r}|y(t)\right)\\&=\int d\mathbf{r}'\left(\widetilde{\psi}_{01}(\mathbf{r}'|y(t))g_{012}^{-1}\left(\mathbf{r}',\mathbf{r}|y(t)\right)+\widetilde{\psi}_{02}(\mathbf{r}'|y(t))g_{022}^{-1}\left(\mathbf{r}',\mathbf{r}|y(t)\right)\right)\\&=-\int d\mathbf{r}'\left(\frac{\delta\beta\left(\mathbf{r}\right)}{\delta\overline{n}_0\left(\mathbf{r}'\right)}\widetilde{n}(\mathbf{r}'|y(t))+\frac{\delta\beta\left(\mathbf{r}'\right)}{\delta\overline{e}_0\left(\mathbf{r}\right)}\widetilde{e}_0(\mathbf{r}'|y(t))\right)\end{aligned} \tag{130}$$

and

$$\begin{aligned}\widetilde{\sigma}_{0ij}(\mathbf{r}|y(t))=&\widetilde{\phi}_{0ij}(\mathbf{r}|y(t))-\beta^{-1}(\mathbf{r},t)\Psi_{02}(\mathbf{r}|y(t))\overline{t}_{ij}^*\left(\mathbf{r},t|y\right)\\=&\widetilde{t}_{0ij}(\mathbf{r}|y(t))-\int d\mathbf{r}'\int d\mathbf{r}''\;\widetilde{\psi}_{0\sigma}(\mathbf{r}'|y(t))g_{0\sigma\alpha}^{-1}\left(\mathbf{r}',\mathbf{r}''\mid y(t)\right)\left.\overline{\psi_{0\alpha}(\mathbf{r}'')\widetilde{t}_{0ij}(\mathbf{r}|y(t))}^{\ell}\right|_{y(t)}\\&-\beta^{-1}(\mathbf{r},t)\Psi_{02}(\mathbf{r}|y(t))\overline{t}_{ij}^*\left(\mathbf{r},t|y\right).\end{aligned} \tag{131}$$

To evaluate the second term on the right note that

$$\begin{aligned}\left.\overline{\psi_{0\alpha}(\mathbf{r}'')\widetilde{t}_{0ij}(\mathbf{r}|y(t))}^{\ell}\right|_{y(t)}&=\left.\overline{t_{0ij}(\mathbf{r})\widetilde{\psi}_{0\alpha}\left(\mathbf{r}''|y(t)\right)}^{\ell}\right|_{y(t)}=-\frac{\delta\left.\overline{t_{0ij}(\mathbf{r})}^{\ell}\right|_{y(t)}}{\delta y_{0\alpha}\left(\mathbf{r}''\right)}\\&=-\frac{\delta\pi_{ij}(\mathbf{r}|y(t))}{\delta y_{0\alpha}\left(\mathbf{r}'',t\right)},\end{aligned} \tag{132}$$

where π_{ij} is the pressure (see Eq. (60) in the main text), so that

$$\begin{aligned}\widetilde{\sigma}_{0ij}(\mathbf{r}|y(t))=&\widetilde{t}_{0ij}(\mathbf{r}|y(t))+\int d\mathbf{r}'\int d\mathbf{r}''\;\widetilde{\psi}_{0\sigma}(\mathbf{r}'|y(t))g_{0\sigma\alpha}^{-1}\left(\mathbf{r}',\mathbf{r}''\mid y(t)\right)\frac{\delta\pi_{ij}(\mathbf{r}|y(t))}{\delta y_0(\mathbf{r}'',t)}\\&-\beta^{-1}(\mathbf{r},t)\Psi_{02}(\mathbf{r}|y(t))\overline{t}_{ij}^*\left(\mathbf{r},t|y\right).\end{aligned} \tag{133}$$

The second term simplifies

$$\int d\mathbf{r}''\widetilde{\psi}_{0\sigma}(\mathbf{r}'|y(t))g^{-1}_{0\sigma\alpha}\left(\mathbf{r}',\mathbf{r}''\mid y(t)\right)\frac{\delta\pi_{ij}(\mathbf{r}|y(t))}{\delta y_0(\mathbf{r}''',t)}$$
$$=-\,\widetilde{n}(\mathbf{r}'|y(t))\int d\mathbf{r}''\left(-g^{-1}_{011}\left(\mathbf{r}',\mathbf{r}''\mid y(t)\right)\frac{\delta\pi_{ij}(\mathbf{r}|y(t))}{\delta\nu\left(\mathbf{r}''|y(t)\right)}\big|_\beta+g^{-1}_{012}\left(\mathbf{r}',\mathbf{r}''\mid y(t)\right)\frac{\delta\pi_{ij}(\mathbf{r}|y(t))}{\delta\beta\left(\mathbf{r}''|y(t)\right)}\big|_\nu\right)$$
$$-\,\widetilde{e}_0(\mathbf{r}'|y(t))\int d\mathbf{r}''\left(-g^{-1}_{021}\left(\mathbf{r}',\mathbf{r}''\mid y(t)\right)\frac{\delta\pi_{ij}(\mathbf{r}|y(t))}{\delta\nu\left(\mathbf{r}''|y(t)\right)}\big|_\beta+g^{-1}_{022}\left(\mathbf{r}',\mathbf{r}''\mid y(t)\right)\frac{\delta\pi_{ij}(\mathbf{r}|y(t))}{\delta\beta\left(\mathbf{r}''|y(t)\right)}\big|_\nu\right)$$
$$=-\,\widetilde{n}(\mathbf{r}'|y(t))\int d\mathbf{r}''\left(-g^{-1}_{011}\left(\mathbf{r}'',\mathbf{r}'\mid y(t)\right)\frac{\delta\pi_{ij}(\mathbf{r}|y(t))}{\delta\nu\left(\mathbf{r}''|y(t)\right)}\big|_\beta+g^{-1}_{021}\left(\mathbf{r}'',\mathbf{r}'\mid y(t)\right)\frac{\delta\pi_{ij}(\mathbf{r}|y(t))}{\delta\beta\left(\mathbf{r}''|y(t)\right)}\big|_\nu\right)$$
$$-\,\widetilde{e}_0(\mathbf{r}'|y(t))\int d\mathbf{r}''\left(-g^{-1}_{012}\left(\mathbf{r}'',\mathbf{r}'\mid y(t)\right)\frac{\delta\pi_{ij}(\mathbf{r}|y(t))}{\delta\nu\left(\mathbf{r}''|y(t)\right)}\big|_\beta+g^{-1}_{022}\left(\mathbf{r}'',\mathbf{r}'\mid y(t)\right)\frac{\delta\pi_{ij}(\mathbf{r}|y(t))}{\delta\beta\left(\mathbf{r}''|y(t)\right)}\big|_\nu\right).\quad(134)$$

In this last equality use has been made of $g^{-1}_{0\alpha\beta}\left(\mathbf{r},\mathbf{r}'\mid y(t)\right)=g^{-1}_{0\beta\alpha}\left(\mathbf{r}',\mathbf{r}\mid y(t)\right)$. Finally, then

$$\widetilde{\psi}_{0\sigma}(\mathbf{r}'|y(t))\int d\mathbf{r}''g^{-1}_{0\sigma\alpha}\left(\mathbf{r}',\mathbf{r}''\mid y(t)\right)\frac{\delta\pi_{ij}(\mathbf{r}|y(t))}{\delta y_0(\mathbf{r}'',t)}$$
$$=-\widetilde{n}(\mathbf{r}'|y(t))\int d\mathbf{r}''\left(-\frac{\delta\pi_{ij}(\mathbf{r}|y(t))}{\delta\nu\left(\mathbf{r}''|y(t)\right)}\big|_\beta\frac{\delta\nu\left(\mathbf{r}''|y(t)\right)}{\delta\overline{n}\left(\mathbf{r}'|y(t)\right)}\mid_e-\frac{\delta\pi_{ij}(\mathbf{r}|y(t))}{\delta\beta\left(\mathbf{r}''|y(t)\right)}\big|_\nu\frac{\delta\beta\left(\mathbf{r}''|y(t)\right)}{\delta\overline{n}\left(\mathbf{r}'|y(t)\right)}\mid_e\right)$$
$$-\,\widetilde{e}_0(\mathbf{r}'|y(t))\int d\mathbf{r}''\left(-\frac{\delta\pi_{ij}(\mathbf{r}|y(t))}{\delta\nu\left(\mathbf{r}''|y(t)\right)}\big|_\beta\frac{\delta\nu\left(\mathbf{r}''|y(t)\right)}{\delta\overline{e}\left(\mathbf{r}'|y(t)\right)}\mid_n-\frac{\delta\pi_{ij}(\mathbf{r}|y(t))}{\delta\beta\left(\mathbf{r}''|y(t)\right)}\big|_\nu\frac{\delta\beta\left(\mathbf{r}''|y(t)\right)}{\delta\overline{e}\left(\mathbf{r}'|y(t)\right)}\mid_n\right)$$
$$=\widetilde{n}(\mathbf{r}'|y(t))\frac{\delta\pi_{ij}(\mathbf{r}|y(t))}{\delta n\left(\mathbf{r}'|y(t)\right)}\mid_e+\widetilde{e}_0(\mathbf{r}'|y(t))\frac{\delta\pi_{ij}(\mathbf{r}|y(t))}{\delta e\left(\mathbf{r}'|y(t)\right)}\mid_n.$$

With this result (133) becomes

$$\widetilde{\sigma}_{0ij}(\mathbf{r}|y(t))=\widetilde{t}_{0ij}(\mathbf{r}|y(t))+\int d\mathbf{r}'\left(\widetilde{n}(\mathbf{r}'|y(t))\frac{\delta\pi_{ij}(\mathbf{r}|y(t))}{\delta n\left(\mathbf{r}'|y(t)\right)}\mid_e+\widetilde{e}_0(\mathbf{r}'|y(t))\frac{\delta\pi_{ij}(\mathbf{r}|y(t))}{\delta e\left(\mathbf{r}'|y(t)\right)}\mid_n\right)$$
$$-\beta^{-1}(\mathbf{r},t)\Psi_{02}(\mathbf{r}|y(t))\overline{t}^*_{ij}\left(\mathbf{r},t|\,y\right).\quad(135)$$

VI. IRREVERSIBLE FLUXES IN THE REST FRAME

The irreversible contributions to the energy and momentum fluxes are defined by

$$\overline{\mathbf{s}}^*_0(\mathbf{r},t|y)\equiv\sum_N\mathrm{Tr}^{(N)}\mathbf{s}_0(\mathbf{r})\Delta_N\left(t\right),\qquad\overline{t}^*_{0ij}(\mathbf{r},t|y)\equiv\sum_N\mathrm{Tr}^{(N)}t_{0ij}(\mathbf{r})\Delta_N\left(t\right)\quad(136)$$

where $\mathbf{s}_0(\mathbf{r})$ and $t_{0ij}(\mathbf{r})$ are the operators for the energy and momentum fluxes in the local rest frame. With (127) these are

$$\overline{\mathbf{s}}^*_0(\mathbf{r},t|y)=\int_0^t dt'\int d\mathbf{r}'\left[\overline{\left(e^{\mathcal{L}(t-t')}\mathbf{s}_0(\mathbf{r})\right)\widetilde{\phi}_{0i2}(\mathbf{r}'|y(t'))}^{\ell}\Big|_{y(t')}\frac{\partial}{\partial r_i}\beta(\mathbf{r}',t')\right.$$
$$-\overline{\left(e^{\mathcal{L}(t-t')}\mathbf{s}_0(\mathbf{r})\right)\widetilde{\sigma}_{0ij}(\mathbf{r}|y(t'))}^{\ell}\Big|_{y(t')}\beta(\mathbf{r}',t')\frac{\partial}{\partial r_j}u_i(\mathbf{r}',t')$$
$$+\overline{\left(e^{\mathcal{L}(t-t')}\mathbf{s}_0(\mathbf{r})\right)\Psi_{02}(\mathbf{r},t')}^{\ell}\Big|_{y(t')}\nabla\cdot\overline{\mathbf{s}}^*_0(\mathbf{r},t'|y)$$
$$\left.+\overline{\left(e^{\mathcal{L}(t-t')}\mathbf{s}_0(\mathbf{r})\right)\Psi_{0j}(\mathbf{r},t')}^{\ell}\Big|_{y(t')}\right]\frac{\partial}{\partial r_i}\overline{t}^*_{ij}(\mathbf{r},t|y)\quad(137)$$

and

$$
\begin{aligned}
\bar{t}^{*}_{0ij}(\mathbf{r},t|y) = \int_0^t dt' \int d\mathbf{r}' \Bigg[& \overline{\left(e^{\mathcal{L}(t-t')} t_{0ij}(\mathbf{r})\right) \widetilde{\phi}_{0k2}(\mathbf{r}'|y(t'))}^{\ell}\Big|_{y(t')} \frac{\partial}{\partial r_k}\beta(\mathbf{r}',t') \\
& - \overline{\left(e^{\mathcal{L}(t-t')} t_{0ij}(\mathbf{r})\right) \widetilde{\sigma}_{0kl}(\mathbf{r}|y(t'))}^{\ell}\Big|_{y(t')} \beta(\mathbf{r}',t') \frac{\partial}{\partial r_k} u_l(\mathbf{r}',t') \\
& + \overline{\left(e^{\mathcal{L}(t-t')} t_{0ij}(\mathbf{r})\right) \Psi_{02}(\mathbf{r}|y(t'))}^{\ell}\Big|_{y(t')} \nabla \cdot \bar{\mathbf{s}}^{*}_{0}(\mathbf{r},t'|y) \\
& + \overline{\left(e^{\mathcal{L}(t-t')} t_{0ij}(\mathbf{r})\right) \Psi_{0l}(\mathbf{r}|y(t'))}^{\ell}\Big|_{y(t')} \Bigg] \frac{\partial}{\partial r_k} \bar{t}^{*}_{kl}(\mathbf{r},t|y).
\end{aligned} \tag{138}
$$

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