Relationships Among Various Forms of Fermi-Dirac Integrals of Low Order

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Different integral forms all are called Fermi-Dirac integrals. The relationship between two such forms is worked out here.

I. DIFFERING DEFNITIONS

Since 1938 at least[1], the basic Fermi-Dirac integral has been given in the form[2,3]

$$I_\alpha(\eta) := \int_0^\infty dx \frac{x^\alpha}{1+\exp(x-\eta)}\ , \quad \alpha > -1 \tag{1}$$

$$I_{\alpha-1}(\eta) = \frac{1}{\alpha}\frac{d}{d\eta} I_\alpha(\eta). \tag{2}$$

This is the form which Karasiev, Sjostrom, and I used in our paper on finite-T generalized gradient approximations for the non-interacting free energy[4].

Blakemore[5] starts with the same definition but calls it F_j. He attributes the definition (1) to Sommerfeld in 1928. Blakemore then introduces what I call F_α, namely

$$F_\alpha(\eta) := \frac{1}{\Gamma(\alpha+1)} I_\alpha(\eta)\ . \tag{3}$$

Huang[6], on the other hand, introduces

$$f_{3/2}(x) = \frac{4}{\sqrt{\pi}}\int_0^\infty dy \frac{y^2}{x^{-1}e^{y^2+1}}\ . \tag{4}$$

This is the form that Jim Dufty is accustomed to using.

Obviously the two forms (Blakemore, Huang) are related but a little care is needed to make sure that expansions given by Blakemore for F_α are written correctly when the variables are changed appropriately to the Huang form.

II. VARIABLE CHANGES

In (4), let

$$x^{-1} = e^{-w} \tag{5}$$

so

$$f_{3/2}(e^w) = \frac{4}{\sqrt{\pi}}\int_0^\infty dy \frac{y^2}{e^{y^2-w}+1}\ . \tag{6}$$

Then let $u = y^2$ so that

$$\begin{aligned} f_{3/2}(e^w) &= \frac{4}{\sqrt{\pi}}\int_0^\infty \frac{du}{2}\frac{u^{1/2}}{e^{u-w}+1} \\ &= \frac{2}{\sqrt{\pi}}\int_0^\infty \frac{du\, u^{1/2}}{e^{u-w}+1} \\ &= F_{1/2}(w). \end{aligned} \tag{7}$$

Or obviously

$$f_{3/2}(w) = F_{1/2}(\ln w)\ . \tag{8}$$

III. EXPANSIONS

At Blakemore's Eq. (25), he reports a fit due to Aymerich-Humet *et al.*[7]. It reads

$$F_\alpha(\eta) = \left[e^{-\eta} + \xi_\alpha(\eta)\right]^{-1}\ . \tag{9}$$

For $\alpha = 1/2$, the function ξ is given by

$$\xi_{1/2}(\eta) = 3\sqrt{\frac{\pi}{2}}\left[(\eta+2.13) + (|\eta-2.13|^{2.4}+9.6)^{5/12}\right]^{-3/2}\ . \tag{10}$$

For use in terms of $f_{3/2}$, eq. (8) with eq. (9) yields

$$f_{3/2}(\eta) = \left[\eta^{-1} + \xi_{1/2}(\ln\eta)\right]^{-1}\ . \tag{11}$$

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