Supporting Information for

Local pressure for inhomogeneous fluids

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S.1. CONSERVATION OF NUMBER AND MOMENTUM DENSITIES

The dynamics of operators is defined by

$$\partial_t X(t) = i\left[\mathcal{H}_N(t), X(t)\right], \tag{S.1}$$

with the Hamiltonian given by

$$\mathcal{H}_N(t) = H_N(t) + \sum_{\alpha=1}^{N} v^{\text{ext}}(\mathbf{q}_\alpha(t), t)), \tag{S.2}$$

$$H_N(t) = \sum_{\alpha=1}^{N} \frac{p_\alpha^2(t)}{2m} + \frac{1}{2}\sum_{\alpha\neq\sigma=1}^{N} U_N(|\mathbf{q}_\alpha(t) - \mathbf{q}_\sigma(t)|). \tag{S.3}$$

The operators corresponding to the number and momentum densities are

$$n(\mathbf{r},t) = \sum_{\alpha=1}^{N} \delta(\mathbf{r} - \mathbf{q}_\alpha(t)), \qquad \mathbf{p}(\mathbf{r},t) = \sum_{\alpha=1}^{N} \frac{1}{2}\left[\mathbf{p}_\alpha(t), \delta(\mathbf{r} - \mathbf{q}_\alpha(t))\right]_+. \tag{S.4}$$

The brackets $[A,B]_+$with a subscript + denote an anti-commutator.

Local conservation laws follow exactly from this Hamiltonian dynamics[1]. The simplest is the conservation of number density

$$\begin{aligned}
\partial_t n(\mathbf{r},t) = i\left[\mathcal{H}_N(t), n(\mathbf{r},t)\right] &= \sum_{\alpha=1}^{N} i\left[\frac{p_\alpha^2(t)}{2m}, \delta(\mathbf{r} - \mathbf{q}_\alpha(t))\right] \\
&= \sum_{\alpha=1}^{N} i\left\{\frac{p_{\alpha j}(t)}{2m}\left[p_{\alpha j}(t), \delta(\mathbf{r} - \mathbf{q}_\alpha(t))\right] + \left[p_{\alpha j}(t), \delta(\mathbf{r} - \mathbf{q}_\alpha(t))\right]\frac{p_{\alpha j}(t)}{2m}\right\} \\
&= \frac{1}{2m}\sum_{\alpha=1}^{N}\left\{p_{\alpha j}(t)\,\partial_{q_{\alpha j}(t)}\delta(\mathbf{r} - \mathbf{q}_\alpha(t)) + \partial_{q_{\alpha j}(t)}\delta(\mathbf{r} - \mathbf{q}_\alpha(t))\,p_{\alpha j}(t)\right\}
\end{aligned} \tag{S.5}$$

or, with the definition of $\mathbf{p}(\mathbf{r})$ in (S.4), the microscopic continuity equation is obtained

$$\partial_t n(\mathbf{r},t) + \frac{1}{m}\nabla\cdot\mathbf{p}(\mathbf{r},t) = 0. \tag{S.6}$$

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Use has been made of the property

$$i\left[p_{\alpha j}\left(t\right),A\left(\mathbf{q}_{\alpha}\left(t\right)\right)\right]=\partial_{q_{\alpha j}\left(t\right)}A\left(\mathbf{q}_{\alpha}\left(t\right)\right). \tag{S.7}$$

The time derivative of $\mathbf{p}\left(\mathbf{r},t\right)$ follows in a similar way

$$\begin{aligned}\partial_{t}p_{j}\left(\mathbf{r},t\right)=&\frac{1}{2}\sum_{\alpha=1}^{N}\left\{\left[p_{\alpha j}\left(t\right),i\left[\mathcal{H}_{N}\left(t\right),\delta\left(\mathbf{r}-\mathbf{q}_{\alpha}\left(t\right)\right)\right]\right]_{+}+\left[i\left[\mathcal{H}_{N}\left(t\right),p_{\alpha j}\left(t\right)\right],\delta\left(\mathbf{r}-\mathbf{q}_{\alpha}\left(t\right)\right)\right]_{+}\right\}\\=&\frac{1}{2}\sum_{\alpha=1}^{N}\left\{\left[p_{\alpha j}\left(t\right),i\left[\frac{p_{\alpha}^{2}\left(t\right)}{2m},\delta\left(\mathbf{r}-\mathbf{q}_{\alpha}\left(t\right)\right)\right]\right]_{+}+\left[i\left[\mathcal{H}_{N}\left(t\right),p_{\alpha j}\left(t\right)\right],\delta\left(\mathbf{r}-\mathbf{q}_{\alpha}\left(t\right)\right)\right]_{+}\right\}\\=&\frac{1}{4m}\sum_{\alpha=1}^{N}\left[p_{\alpha j}\left(t\right),\left[p_{\alpha k}\left(t\right),\partial_{q_{\alpha k}\left(t\right)}\Delta\left(\mathbf{r}-\mathbf{q}_{\alpha}\left(t\right)\right)\right]_{+}\right]_{+}-\frac{1}{2}\sum_{\alpha=1}^{N}\delta\left(\mathbf{r}-\mathbf{q}_{\alpha}\left(t\right)\right)\partial_{q_{\alpha j}\left(t\right)}v^{\text{ext}}(\mathbf{q}_{\alpha}\left(t\right),t))\\&-\frac{1}{4}\sum_{\alpha=1}^{N}\sum_{\beta,\sigma=1}^{N}\left[\delta\left(\mathbf{r}-\mathbf{q}_{\alpha}\left(t\right)\right),\partial_{q_{\alpha j}\left(t\right)}V\left(\left|\mathbf{q}_{\beta}\left(t\right)-\mathbf{q}_{\sigma}\left(t\right)\right|\right)\right]_{+}\\[1em]=&-\partial_{r_{k}}\frac{1}{4m}\sum_{\alpha=1}^{N}\left[p_{\alpha j}\left(t\right),\left[p_{\alpha k}\left(t\right),\delta\left(\mathbf{r}-\mathbf{q}_{\alpha}\left(t\right)\right)\right]_{+}\right]_{+}+\frac{1}{2}\sum_{\alpha=1}^{N}\delta\left(\mathbf{r}-\mathbf{q}_{\alpha}\left(t\right)\right)F_{\alpha j}^{\text{ext}}\left(\mathbf{q}_{\alpha}\left(t\right)\right)\\&+\frac{1}{2}\sum_{\alpha=1}^{N}\sum_{\beta=1}^{N}F_{\alpha\beta j}\left(\left|\mathbf{q}_{\alpha}\left(t\right)-\mathbf{q}_{\beta}\left(t\right)\right|\right)\delta\left(\mathbf{r}-\mathbf{q}_{\alpha}\left(t\right)\right)\end{aligned} \tag{S.8}$$

where $F_{\alpha\beta j}\left(\left|\mathbf{q}_{\alpha}\left(t\right)-\mathbf{q}_{\beta}\left(t\right)\right|\right)$ is the j^{th} component of the force on particle α due to particle β

$$F_{\alpha\beta j}\left(\left|\mathbf{q}_{\alpha}-\mathbf{q}_{\beta}\right|\right)=-\partial_{q_{\alpha j}\left(t\right)}V\left(\left|\mathbf{q}_{\alpha}\left(t\right)-\mathbf{q}_{\beta}\left(t\right)\right|\right), \tag{S.9}$$

and $F_{\alpha j}^{\text{ext}}\left(\mathbf{q}_{\alpha}\left(t\right)\right)$ is the j^{th} component of the force on particle α due to the external potential

$$F_{\alpha j}^{\text{ext}}\left(\mathbf{q}_{j}\right)=-\partial_{q_{\alpha j}\left(t\right)}u\left(\mathbf{q}_{\alpha}\left(t\right),t\right). \tag{S.10}$$

The last term on the right side of (S.8) can be rewritten as

$$\begin{aligned}\sum_{\alpha\neq\beta=1}^{N}F_{\alpha\beta j}\left(\left|\mathbf{q}_{\alpha}\left(t\right)-\mathbf{q}_{\beta}\left(t\right)\right|\right)\delta\left(\mathbf{r}-\mathbf{q}_{\alpha}\left(t\right)\right)=&\sum_{\alpha\neq\beta=1}^{N}F_{\beta\alpha j}\left(\left|\mathbf{q}_{\beta}\left(t\right)-\mathbf{q}_{\alpha}\left(t\right)\right|\right)\delta\left(\mathbf{r}-\mathbf{q}_{\beta}\left(t\right)\right)\\=&-\sum_{\alpha\neq\beta=1}^{N}F_{\alpha\beta j}\left(\left|\mathbf{q}_{\alpha}\left(t\right)-\mathbf{q}_{\beta}\left(t\right)\right|\right)\delta\left(\mathbf{r}-\mathbf{q}_{\beta}\left(t\right)\right)\end{aligned} \tag{S.11}$$

as follows from Newton's third law. Therefore

$$\begin{aligned}\sum_{\alpha\neq\beta=1}^{N}F_{\alpha\beta j}\left(\left|\mathbf{q}_{\alpha}\left(t\right)-\mathbf{q}_{\beta}\left(t\right)\right|\right)\delta\left(\mathbf{r}-\mathbf{q}_{\alpha}\left(t\right)\right)=&\frac{1}{2}\sum_{\alpha\neq\beta=1}^{N}F_{\alpha\beta j}\left(\left|\mathbf{q}_{\alpha}\left(t\right)-\mathbf{q}_{\beta}\left(t\right)\right|\right)\\&\times\left(\delta\left(\mathbf{r}-\mathbf{q}_{\alpha}\left(t\right)\right)-\delta\left(\mathbf{r}-\mathbf{q}_{\beta}\left(t\right)\right)\right).\end{aligned} \tag{S.12}$$

Next note the identity

$$\delta\left(\mathbf{r}-\mathbf{q}_{1}\right)-\delta\left(\mathbf{r}-\mathbf{q}_{2}\right)=\int_{\lambda_{2}}^{\lambda_{1}}d\lambda\frac{d}{d\lambda}\delta\left(\mathbf{r}-\mathbf{x}\left(\lambda\right)\right),\qquad\mathbf{x}\left(\lambda_{1}\right)=\mathbf{q}_{1},\qquad\mathbf{x}\left(\lambda_{2}\right)=\mathbf{q}_{2}. \tag{S.13}$$

Here $\mathbf{x}\left(\lambda\right)$ is an arbitrary path for the vector $\mathbf{x}$ moving from $\mathbf{x}\left(\lambda_{2}\right)=\mathbf{q}_{2}$ to $\mathbf{x}\left(\lambda_{1}\right)=\mathbf{q}_{1}$. Carrying out the derivative gives

$$\delta\left(\mathbf{r}-\mathbf{q}_{1}\right)-\delta\left(\mathbf{r}-\mathbf{q}_{2}\right)=-\partial_{r_{k}}\mathcal{D}_{k}\left(\mathbf{r},\mathbf{q}_{1},\mathbf{q}_{2}\right), \tag{S.14}$$

where

$$\mathcal{D}_k\left(\mathbf{r},\mathbf{q}_1,\mathbf{q}_2\right)=\int_{\lambda_2}^{\lambda_1}d\lambda\frac{dx_k\left(\lambda\right)}{d\lambda}\delta\left(\mathbf{r}-\mathbf{x}\left(\lambda\right)\right). \tag{S.15}$$

Use of (S.14) in (S.12) and (S.8) gives the momentum conservation law

$$\partial_t p_j\left(\mathbf{r},t\right)+\partial_{r_k}t_{jk}\left(\mathbf{r},t\right)=f_j\left(\mathbf{r},t\right) \tag{S.16}$$

where the total momentum flux $t_{\alpha\beta}\left(\mathbf{r},t\right)$ is the sum of a kinetic and potential part

$$t_{\alpha\beta}\left(\mathbf{r},t\right)=t_{\alpha\beta}^{K}\left(\mathbf{r},t\right)+t_{\alpha\beta}^{P}\left(\mathbf{r},t\right), \tag{S.17}$$

$$t_{jk}^{K}\left(\mathbf{r},t\right)=\frac{1}{4m}\sum_{\alpha=1}^{N}\left[p_{\alpha j}\left(t\right),\left[p_{\alpha k}\left(t\right),\delta\left(\mathbf{r}-\mathbf{q}_\alpha\left(t\right)\right)\right]_+\right]_+, \tag{S.18}$$

$$t_{jk}^{P}\left(\mathbf{r},t\right)=\frac{1}{4}\sum_{\alpha\neq\beta=1}^{N}F_{\alpha\beta j}\left(\left|\mathbf{q}_\alpha\left(t\right)-\mathbf{q}_\beta\left(t\right)\right|\right)\mathcal{D}_k\left(\mathbf{r},\mathbf{q}_\alpha\left(t\right),\mathbf{q}_\beta\left(t\right)\right). \tag{S.19}$$

The right side of (S.16) is the force density of the external potential

$$f_j\left(\mathbf{r},t\right)=\sum_{\alpha=1}^{N}\delta\left(\mathbf{r}-\mathbf{q}_j\left(t\right)\right)F_{\alpha j}^{\text{ext}}\left(\mathbf{q}_\alpha\left(t\right)\right). \tag{S.20}$$

S.2. DETERMINATION OF MOMENTUM FLUX

The straightforward derivation of the momentum conservation law in Appendix S.1 leads to a momentum flux (as in the text the dependence on t will be left implicit as it plays no role in the following)

$$\begin{aligned}t_{ij}\left(\mathbf{r}\right)=&\frac{1}{4m}\sum_{\alpha=1}^{N}\left[p_{i\alpha},\left[p_{j\alpha},\delta\left(\mathbf{r}-\mathbf{q}_\alpha\right)\right]_+\right]_+\\&+\frac{1}{2}\sum_{\alpha\neq\sigma=1}^{N}F_{\alpha\sigma i}\left(\left|\mathbf{q}_\alpha-\mathbf{q}_\sigma\right|\right)\mathcal{D}_j\left(\mathbf{r},\mathbf{q}_\alpha,\mathbf{q}_\sigma\right),\end{aligned} \tag{S.21}$$

where the operator $\mathcal{D}_j\left(\mathbf{r},\mathbf{q}_\alpha,\mathbf{q}_\sigma\right)$ is given by

$$\mathcal{D}_j\left(\mathbf{r},\mathbf{q}_1,\mathbf{q}_2\right)\equiv\int_{\mathcal{C}}d\lambda\frac{dx_j\left(\lambda\right)}{d\lambda}\delta\left(\mathbf{r}-\mathbf{x}\left(\lambda\right)\right),\qquad\mathbf{x}\left(\lambda_1\right)=\mathbf{q}_1,\qquad\mathbf{x}\left(\lambda_2\right)=\mathbf{q}_2. \tag{S.22}$$

Here $\mathcal{C}$ is an arbitrary continuous path connecting $\mathbf{x}\left(\lambda\right)$ between λ_1 and λ_2. Consider the simplest choice a linear path

$$\mathbf{x}\left(\lambda\right)=\mathbf{q}_1+\left(\mathbf{q}_2-\mathbf{q}_1\right)\lambda. \tag{S.23}$$

Then

$$\mathcal{D}_j\left(\mathbf{r},\mathbf{q}_1,\mathbf{q}_2\right)\rightarrow\left(q_{2j}-q_{1j}\right)\int_0^1 d\lambda\delta\left(\mathbf{r}-\mathbf{q_1}-\left(\mathbf{q_2}-\mathbf{q_1}\right)\lambda\right),\qquad\mathbf{x}\left(\lambda_1\right)=\mathbf{q}_1,\qquad\mathbf{x}\left(\lambda_2\right)=\mathbf{q}_2. \tag{S.24}$$

Since $F_{\alpha\sigma i}\left(\left|\mathbf{q}_\alpha-\mathbf{q}_\sigma\right|\right)\propto\left(q_{2i}-q_{1i}\right)$ use of (S.24) in (S.21) leads to a form for the momentum flux that is symmetric, $t_{ij}\left(\mathbf{r}\right)=t_{ji}\left(\mathbf{r}\right)$, and consequently its divergence with respect to first and second indices are the same

$$\partial_j t_{ij}\left(\mathbf{r}\right)=\partial_j t_{ji}\left(\mathbf{r}\right). \tag{S.25}$$

Next define another symmetric momentum flux whose divergences are the same as those of (S.25)

$$t'_{ij}(\mathbf{r}) = t_{ij}(\mathbf{r}) + \epsilon_{ik\ell}\epsilon_{jmn}\partial_k\partial_m A_{\ell n}(\mathbf{r}), \tag{S.26}$$

for some unspecified tensor field $A_{\ell n}(\mathbf{r})$ and $\epsilon_{ik\ell}$ is the Levi-Cevita tensor. This added term is seen to be the curl of a vector associated with each component of $t_{ij}(\mathbf{r})$ (i.e., noting that $t_{ij}(\mathbf{r})$ transforms as vector for components i at fixed j, and vice versa). With the additional condition $A_{\ell n}(\mathbf{r}) = A_{n\ell}(\mathbf{r})$, it is verified that

$$t'_{ij}(\mathbf{r}) = t'_{ji}(\mathbf{r}), \tag{S.27}$$

$$\partial_j t'_{ij}(\mathbf{r}) = \partial_j t'_{ji}(\mathbf{r}). \tag{S.28}$$

The two indefinite features of the momentum flux have now been made explicit with the contour fixed by symmetry and the choice of a term with vanishing divergence. It is interesting to note that the contribution from different contours, denoted by $\Delta\mathcal{D}_j(\mathbf{r}, \mathbf{q}_1, \mathbf{q}_2)$ has a vanishing divergence, since the endpoints of the integration in (S.24) are the same for all contours. Consequently, the divergence of $\Delta\mathcal{D}_j$ vanishes, $\partial_j\Delta\mathcal{D}_j = 0$. It follows that $\Delta\mathcal{D}_j$ is the curl of some vector. Hence, the difference between contours is included in the form (S.26).

S.3. PERCUS PRESSURE TENSOR

The free energy is obtained from the grand potential by a Legendre transform

$$\beta F(\beta, V \mid \overline{n}) = -Q^e(\beta, V \mid \nu) + \int d\mathbf{r}\overline{n}(\mathbf{r})\nu(\mathbf{r}) \tag{S.29}$$

where $\overline{n}(\mathbf{r}) = \langle n(\mathbf{r})\rangle$ is the average number density. The free energy and grand potential are expressed as integrals of their respective densities

$$F(\beta, V \mid \overline{n}) = \int d\mathbf{r} f(\mathbf{r}, \beta \mid \overline{n}), \qquad Q^e(\beta, V \mid \nu) = \int d\mathbf{r}\beta p_T(\mathbf{r}, \beta \mid \nu) \tag{S.30}$$

so that

$$\beta p_T(\mathbf{r}, \beta \mid \nu) = \overline{n}(\mathbf{r})\nu(\mathbf{r}) - f(\mathbf{r}, \beta \mid \overline{n}). \tag{S.31}$$

The Percus pressure tensor[2] is defined by

$$p_{ij}(\mathbf{r}, \beta \mid \overline{n}) \equiv \delta_{ij} p_T(\mathbf{r}, \beta \mid \nu) + \int d\mathbf{r}' r'_i \int_0^1 d\gamma \frac{\delta f(\mathbf{r} + \gamma\mathbf{r}' - \mathbf{r}', \beta \mid \overline{n})}{\delta\overline{n}(\mathbf{r} + \gamma\mathbf{r}')} \partial_j \overline{n}(\mathbf{r} + \gamma\mathbf{r}'). \tag{S.32}$$

The corresponding mechanical pressure is

$$p_m(\mathbf{r}, \beta \mid \overline{n}) = p_T(\mathbf{r}, \beta \mid \nu) + \frac{1}{3}\int d\mathbf{r}' \int_0^1 d\gamma \frac{\delta f(\mathbf{r} + \gamma\mathbf{r}' - \mathbf{r}', \beta \mid \overline{n})}{\delta\overline{n}(\mathbf{r} + \gamma\mathbf{r}')} r'_i \partial_i \overline{n}(\mathbf{r} + \gamma\mathbf{r}'). \tag{S.33}$$

No motivation nor interpretation for this result is provided.

First, prove the force balance equation. Separate the pressure tensor into two parts

$$p_{ij}(\mathbf{r}, \beta \mid \overline{n}) = \delta_{ij} p_T(\mathbf{r}, \beta \mid \overline{n}) + \Delta P_{ij}(\mathbf{r}, \beta \mid \overline{n}), \tag{S.34}$$

$$\Delta P_{ij}(\mathbf{r}, \beta \mid \overline{n}) = \int d\mathbf{r}' r'_i \int_0^1 d\gamma \frac{\delta f(\mathbf{r} + \gamma\mathbf{r}' - \mathbf{r}', \beta \mid \overline{n})}{\delta\overline{n}(\mathbf{r} + \gamma\mathbf{r}')} \partial_j \overline{n}(\mathbf{r} + \gamma\mathbf{r}').$$

Then

$$\begin{aligned} \delta_{ij}\partial_i \beta p_T(\mathbf{r}, \beta \mid \overline{n}) &= \delta_{ij}\partial_i(\overline{n}(\mathbf{r})\nu(\mathbf{r}) - \beta f(\mathbf{r}, \beta \mid \overline{n})) \\ &= \overline{n}(\mathbf{r})\partial_j\nu(\mathbf{r}) + \nu(\mathbf{r})\partial_j\overline{n}(\mathbf{r}) - \partial_j\beta f(\mathbf{r}, \beta \mid \overline{n}) \end{aligned} \tag{S.35}$$

and

$$
\begin{aligned}
\partial_i \beta \Delta p_{ij}\left(\mathbf{r},\beta \mid \overline{n}\right) &= \int d\mathbf{r}' r_i' \partial_i \int_0^1 d\gamma \frac{\delta \beta f\left(\mathbf{r}+\gamma \mathbf{r}'-\mathbf{r}',\beta \mid \overline{n}\right)}{\delta \overline{n}(\mathbf{r}+\gamma \mathbf{r}')} \partial_j \overline{n}\left(\mathbf{r}+\gamma \mathbf{r}'\right) \\
&= \int d\mathbf{r}' \int_0^1 d\gamma \partial_\gamma \left(\frac{\delta \beta f\left(\mathbf{r}+\gamma \mathbf{r}'-\mathbf{r}',\beta \mid \overline{n}\right)}{\delta \overline{n}(\mathbf{r}+\gamma \mathbf{r}')} \partial_j \overline{n}\left(\mathbf{r}+\gamma \mathbf{r}'\right) \right) \\
&= \int d\mathbf{r}' \left(\frac{\delta \beta f\left(\mathbf{r},\beta \mid \overline{n}\right)}{\delta \overline{n}(\mathbf{r}+\mathbf{r}')} \partial_j \overline{n}\left(\mathbf{r}+\mathbf{r}'\right) - \frac{\delta \beta f\left(\mathbf{r}-\mathbf{r}',\beta \mid \overline{n}\right)}{\delta \overline{n}(\mathbf{r})} \partial_j \overline{n}\left(\mathbf{r}\right) \right) \\
&= \int d\mathbf{r}_1 \frac{\delta \beta f\left(\mathbf{r},\beta \mid \overline{n}\right)}{\delta \overline{n}(\mathbf{r}_1)} \partial_{1j} \overline{n}\left(\mathbf{r}_1\right) - \frac{\delta \beta F\left(\beta \mid \overline{n}\right)}{\delta \overline{n}(\mathbf{r})} \partial_j \overline{n}\left(\mathbf{r}\right) \\
&= -\left(\nu(\mathbf{r}) \partial_j \overline{n}\left(\mathbf{r}\right) - \partial_j \beta f\left(\mathbf{r},\beta \mid \overline{n}\right)\right)
\end{aligned} \tag{S.36}
$$

so the force balance equation is verified

$$
\partial_i \beta p_{ij}\left(\mathbf{r},\beta \mid \overline{n}\right) = \overline{n}(\mathbf{r}) \partial_j \nu(\mathbf{r}). \tag{S.37}
$$

For the pressure to qualify as a thermodynamic pressure it should satisfy

$$
\int d\mathbf{r} \beta p_m\left(\mathbf{r},\beta \mid \nu\right) = Q^e\left(\beta, V \mid \nu\right). \tag{S.38}
$$

Use of the Percus form (S.33) gives this condition to be

$$
\int d\mathbf{r} \frac{1}{3} \int d\mathbf{r}' \int_0^1 d\gamma \frac{\delta f\left(\mathbf{r}+\gamma \mathbf{r}'-\mathbf{r}',\beta \mid \overline{n}\right)}{\delta \overline{n}(\mathbf{r}+\gamma \mathbf{r}')} r_i' \partial_i \overline{n}\left(\mathbf{r}+\gamma \mathbf{r}'\right) = 0. \tag{S.39}
$$

The Appendix of Ref. 3 claims that this is true, but no proof is given.

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[1] Dufty JW, Luo K, Wrighton JM. Generalized hydrodynamics revisited. Phys Rev Research. 2020;2:023036.
[2] Percus JK. The pressure tensor in a non-uniform fluid. Chem Phys Lett. 1986;123:311–314.
[3] Pozhar LA. Transport theory of inhomogeneous fluids. World Scientific, 1994.